Thom-Sebastiani and Duality for Matrix Factorizations, and Results on the Higher Structures of the Hochschild Invariants

by

Anatoly Preygel

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Abstract

The derived category of a hypersurface has an action by "cohomology operations" $k[\![\beta]\!]$, $\deg \beta = 2$, underlying the 2-periodic structure on its category of singularities (as *matrix factorizations*). We prove a Thom-Sebastiani type Theorem, identifying the $k[\![\beta]\!]$-linear tensor products of these dg categories with coherent complexes on the zero locus of the sum potential on the product (with a support condition), and identify the dg category of colimit-preserving $k[\![\beta]\!]$-linear functors between Ind-completions with Ind-coherent complexes on the zero locus of the difference potential (with a support condition). These results imply the analogous statements for the 2-periodic dg categories of matrix factorizations. We also present a viewpoint on matrix factorizations in terms of (formal) groups actions on categories that is conducive to formulating functorial statements and in particular to the computation of higher algebraic structures on Hochschild invariants. Some applications include: we refine and establish the expected computation of 2-periodic Hochschild invariants of matrix factorizations; we show that the category of matrix factorizations is smooth, and is proper when the critical locus is proper; we show how Calabi-Yau structures on matrix factorizations arise from volume forms on the total space; we establish a version of Knörrer Periodicity for eliminating metabolic quadratic bundles over a base.

Thesis Supervisor: Jacob A. Lurie
Title: Professor of Mathematics, Harvard University

Contents

1 Introduction **5**
- 1.1 Integral transforms for (Ind) coherent complexes 5
- 1.2 Matrix factorizations 6
- 1.3 Summary of results 8

2 Notation and background **11**
- 2.1 Gradings, categories, etc. 11
- 2.2 Primer on $\mathrm{QC}^! = \mathrm{Ind}\,\mathrm{DCoh}$ 15

3 From coherent complexes to matrix factorizations: Derived (based) loop spaces, and first proofs of Theorems **21**
- 3.1 Generalities on PreMF and MF 21
- 3.2 Thom-Sebastiani & duality Theorems for (pre-) matrix factorizations 38

4 Support and completion in DAG, and more matrix factorizations via derived (based) loops **49**
- 4.1 Completion via derived Cech nerve and derived groups 49

5 MF via groups acting on categories: Categorified Cartier/Fourier duality **59**
- 5.1 Introduction 59
- 5.2 Discrete Version: MF via invertible functions and S^1-actions 61
- 5.3 Generalities on Formal Groups acting on Categories 68
- 5.4 Infinitesimal Version: MF via functions and $B\widehat{\mathbb{G}_a}$-actions 76
- 5.5 Comparison of three viewpoints 79

6 Applications: Hochschild invariants (unstructured), smoothness, etc. **83**
- 6.1 Smooth, proper, CY, and **HH** 83
- 6.2 Quadratic bundles 95

7 Adjoint Actions for E_2-algebras and the Hochschild Package of Matrix Factorizations **105**
- 7.1 Introduction 105
- 7.2 E_n-algebras, P_n-algebras, and Calculus-variants 113
- 7.3 Units and adjoint actions for loop spaces, E_k-algebras, etc. 114
- 7.4 Infinitesimal adjoint actions for Lie-, E_n-, P_n-, etc. algebras 119
- 7.5 Making things explicit in the case $k = 2$ 122

A Ind-Coherent Complexes **125**
A.1 Descent for $\mathrm{QC}^!$ and MF^∞ 125
A.2 Integral transforms for (Ind) Coherent Complexes 133

Chapter 1

Introduction

This thesis is a revised and somewhat extended version of [P2]. Our goal is to establish some basic results about tensor products and functor categories between 2-periodic ($=k(\!(\beta)\!)$-linear, $\deg\beta = -2$) dg-categories of matrix factorizations, beyond the case of isolated singularities. These results are surely unsurprising, however our approach may be of interest: Rather than working directly in the 2-periodic or curved contexts, we deduce the results from more refined statements about the $k[\![\beta]\!]$-linear dg-category of coherent sheaves on the special fiber so that we are able to remain in the more familiar world of coherent sheaves. This is done using a convenient (derived) geometric description of the $k[\![\beta]\!]$-linear structure of cohomology operations.

One motivation for this work was establishing certain expected computations of Hochschild invariants of the 2-periodic dg-category of matrix factorizations. In chapter 6 we do this at the level of underlying complexes. In chapter 7 we extend this to some of the richer structures carried by the Hochschild invariants. Specifically, we address the E_2-algebra type structures; though we had planned to also address the "connection"-type structure on periodic cyclic chains that didn't quite make the editting-cut for this thesis.

1.1 Integral transforms for (Ind) coherent complexes

Suppose X is a nice scheme (or derived scheme, stack, formal scheme, etc.). The non-commutative viewpoint tells us to forget X and pass to its "non-commutative" shadow: the dg-category $\mathrm{Perf}(X)$ or its Ind-completion $\mathrm{QC}(X)$. Work of Toën [T2], ..., Ben Zvi-Francis-Nadler [BZFN] provide us with useful tools relating the commutative and non-commutative worlds:

– A "tensor product theorem," stating that (derived) fiber products of schemes go to tensor products of dg-categories:

$$\mathrm{Perf}(X)\otimes_{\mathrm{Perf}(S)}\mathrm{Perf}(Y) \xrightarrow{\sim} \mathrm{Perf}(X\times_S Y) \qquad \mathrm{QC}(X)\widehat{\otimes}_{\mathrm{QC}(S)}\mathrm{QC}(Y) \xrightarrow{\sim} \mathrm{QC}(X\times_S Y)$$

– A description of functor categories: Every quasi-coherent complex on the product gives rise to an "integral transform" functor, and this determines an equivalence

$$\mathrm{QC}(X\times_S Y) \xrightarrow{\sim} \mathrm{Fun}^L_{\mathrm{QC}(S)}(\mathrm{QC}(X), \mathrm{QC}(Y))$$

identifying $\mathrm{QC}(X\times Y)$ with the dg-category $\mathrm{Fun}^L_{\mathrm{QC}(S)}(\mathrm{QC}(X), \mathrm{QC}(Y))$ of colimit-preserving $\mathrm{QC}(S)^{\otimes}$-linear functors $\mathrm{QC}(X) \to \mathrm{QC}(Y)$ (also known as "bimodules"). The identity functor corresponds to $\Delta_* \mathcal{O}_X$ (=the diagonal bimodule), and the trace of endofunctors (=Hochschild homology) corresponds to taking global sections of the pullback along the diagonal. Thus we have descriptions of the functor category and of the Hochschild invariants in familiar commutative terms.

When studying non-smooth schemes X, it's convenient to replace vector bundles by coherent sheaves. Analogously, to replace perfect complexes $\mathrm{Perf}(X)$ by (bounded) coherent complexes $\mathrm{DCoh}(X)$; and, to replace quasi-coherent complexes $\mathrm{QC}(X)$ by the larger $\mathrm{QC}^!(X) = \mathrm{Ind}\,\mathrm{DCoh}(X)$ of *Ind-coherent (aka shriek quasi-coherent) complexes*. Provided we work with finite-type schemes over a perfect base-field, the analogs of the above two theorems remain true: this is essentially the content of Lunts paper [L1].[1] Section A.2 develops the mild extensions which we will need (to derived schemes, and with support conditions) in the more geometric language that we will wish to use: The "tensor product theorem" for DCoh and $\mathrm{QC}^!$ (Prop. A.2.3.2), and a description of functor categories in terms of "shriek" integral transforms (Theorem A.2.2.4).

Two direct applications may be worth highlighting:

- One can write down formulas for the Hochschild invariants of $\mathrm{DCoh}(X)$ for not-necessarily smooth X, and with support conditions (Cor. A.2.5.1). For $\mathbf{HH}_\bullet$, this makes manifest the "Poincaré duality" between what might be called Hochschild K-theory and Hochschild G-theory. For $\mathbf{HH}^\bullet$, one obtains the somewhat strange looking fact that $\mathbf{HH}^\bullet(\mathrm{DCoh}(X)) \simeq \mathbf{HH}^\bullet(\mathrm{Perf}(X))$.
- Obviously one re-obtains Lunts' result that $\mathrm{DCoh}(X)$ is smooth, and one sees that this fails for even very nice formal schemes:[2] $\mathrm{DCoh}_Z(X) = \mathrm{DCoh}(\widehat{X_Z})$ is *not* usually smooth (even when both Z and X are smooth), though the failure of smoothness is in a sense mild (e.g., the identity functor is a uniformly t-bounded filtered colimit of compacts). One consequence is that $\mathbf{HH}_\bullet(\mathrm{DCoh}_Z(X))$ admits a nice description while $\mathbf{HH}^\bullet(\mathrm{DCoh}_Z(X))$ does not.

1.2 Matrix factorizations

Suppose $f\colon M \to \mathbb{A}^1$ is a map from a smooth scheme to $\mathbb{A}^1$, and that we are interested in the geometry of f over a formal disc near the origin. We can attach to it several non-commutative shadows, the two simplest candidates being the dg-categories $\mathrm{DCoh}(M_0)$ and $\mathrm{Perf}(\widehat{M_0}) = \mathrm{Perf}_{M_0}(M)$. However, these both lose too much information: they do not depend on the defining function f, and the second one doesn't even depend on the scheme structure on M_0. A standard way to remedy this is to consider the 2-periodic(=$k((\beta))$-linear, $\deg \beta = -2$) dg-category $\mathrm{MF}(M, f) \simeq \mathrm{DSing}(M_0)$ of "matrix factorizations" or "LG D-branes" (at a *single* critical value).

A starting point for our study is the observation[3] that there are three (essentially equivalent, pairwise Koszul dual) refinements of this. Using f, one can put extra structure

[1] The author originally learned that such a result might be true from Jacob Lurie, who attributed it to conversation with Dennis Gaitsgory. The author wrote up the mild extensions of Section A.2 before finding Lunts' paper and realizing that it proved essentially the same thing.↑

[2] There are several ways one could wish to define $\mathrm{DCoh}(\widehat{X_Z})$: Our choice is as the compact objects in $\mathrm{QC}^!(\widehat{X_Z})$ which is constructed as the ∞-categorical inverse limit along shriek-pullback of $\mathrm{QC}^!$ on nilthickenings of Z. See Theorem 4.1.2.8 for a sketch of the comparison and references.↑

[3] Due in parts to several people, notably Constantin Teleman for the connection to S^1-actions.↑

($k[\![\beta]\!]$-linearity) on $\mathrm{DCoh}(M_0)$ and extra structure (an S^1-action or $B\widehat{\mathbb{G}_a}$-action) on $\mathrm{DCoh}(M)$ or $\mathrm{DCoh}(\widehat{M_0})$:

(i) One can regard $\mathrm{DCoh}(M)$ as linear over $\mathrm{Perf}(\mathbb{A}^1)^{\otimes} = (\mathrm{Perf}\, k[x], \otimes_{k[x]})$. Variant: One can regard $\mathrm{DCoh}(\widehat{M_0}) = \mathrm{DCoh}_{M_0}(M)$ as linear over $\mathrm{Perf}(\widehat{0})^{\otimes} = \mathrm{Perf}_0(\mathbb{A}^1)^{\otimes}$.

(ii) (See Section 3.1.) The 2-periodicity on $\mathrm{DSing}(M_0)$ comes from a $k[\![\beta]\!]$-linear structure on $\mathrm{DCoh}(M_0)$, for which we give a (derived) geometric description in §3.1.1. We call this $k[\![\beta]\!]$-linear dg-category $\mathrm{PreMF}(M, f)$ to emphasize the dependence on f. Despite the "Pre" in the name, $\mathrm{PreMF}(M, f)$ is a refinement of $\mathrm{MF}(M, f)$. We'll see in Cor. 3.1.4.4 and Cor. 3.1.2.4 that $\mathrm{PreMF}(M, f)$ allows one to recover all the other actors in the story:

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} \{\text{locally } \beta\text{-torsion } k[\![\beta]\!]\text{-modules}\} \simeq \mathrm{Perf}(M_0)$$

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)\text{-mod} \simeq \mathrm{MF}(M, f)$$

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k \simeq \mathrm{DCoh}_{M_0}(M)$$

(iii) (See chapter 5.) There is a (*homotopy*) S^1-action on $\mathrm{DCoh}_{M_0}(M) = \mathrm{DCoh}(\widehat{M_0})$. This S^1-action is fundamental, as it allows one to recover the other actors in the story (Cor. 5.2.1.4):[4]

$$\mathrm{DCoh}(\widehat{M_0})_{S^1} \simeq \mathrm{Perf}(M_0)$$

$$\mathrm{DCoh}(\widehat{M_0})^{S^1} \simeq \mathrm{PreMF}(M, f)$$

$$\mathrm{DCoh}(\widehat{M_0})^{\mathrm{Tate}} \simeq \mathrm{MF}(M, f)$$

where these equivalence are $C^*(BS^1) = k[\![\beta]\!]$-linear. *Important Variants:* One can avoid completing along the zero fiber(/imposing support conditions) in two ways: Replacing maps $M \to \mathbb{A}^1$ by maps $M \to \mathbb{G}_m$; or, replacing S^1-actions with $B\widehat{\mathbb{G}}_a$-actions. The latter requires a diversion into the theory of derived formal groups acting on categories.

Remark 1.2.0.1. There are two bits of Philosophy that can help to organize the tangled list above, and that are worth bringing to the fore:

– In chapter 5 we note that for A a finite rank free abelian group (i.e., $\mathbb{Z}^n$) and for V a vector space one has equivalences

$$\mathbf{dgcat}_{/\mathbb{G}_m \otimes A^\vee} \simeq \mathbf{dgcat}_{/B^2 A} \qquad \mathbf{dgcat}_{/\mathbb{G}_a \otimes V^\vee} \simeq \mathbf{dgcat}_{/B^2 \widehat{V}}.$$

The left hand side should be thought of as a higher form of Cartier duality, and the right hand side as a higher form of Fourier duality. In each case, one was classically relating functions or locally constant sheaves on two "dual" abelian groups things.

Setting $A = \mathbb{Z}$ and $V = k$, let us spell out what these statements are saying. On the left, it is saying that making something $k[x, x^{-1}]$-linear (e.g., a map from a scheme to $\mathbb{G}_m$) is the same as giving it an action of the simplicial group $S^1 = B\mathbb{Z}$. Under this, invariants and coinvariants will go to two different notions of "the fiber over 1"

[4] It *is* important in the following formulas that we passed to compact objects: Taking invariants does not commute with forming Ind categories.↑

that exist; there's a map between them, and the quotient is the Tate construction that comes up so much. On the right, it is saying that making something $k[x]$-linear (e.g., a map from a scheme to $\mathbb{G}_a$) is the same as giving it an action of the derived formal group $B\widehat{\mathbb{G}_a}$.

Under this, invariants and coinvariants will go to two different notions of "the fiber over 0" that exist; there's a map between them, and the quotient is the Tate construction that comes up so much. These two senses of taking fiber can be thought of as taking "star" and "shriek" pullbacks of small dg-categories, where the "star" one takes Perf $\mapsto$ Perf, and the "shriek" one–at least along closed immersions, like the inclusion of a fiber–takes DCoh $\mapsto$ DCoh. This gives one explanation why Perf and DCoh of the fiber should be $k[\![\beta]\!]$-linear: $k[\![\beta]\!]$ is the "ring of functions" on $B^2\widehat{\mathbb{G}_a}$ or $B^2\mathbb{Z}$.

– In chapter 3 and chapter 4 we take a slightly less categorical tack. The starting point there is the following geometric incarnation of Koszul duality: If one has a map $f\colon E \to B$, and one wants to study it in a neighborhood of the fiber over $b \in B$, then just taking the fiber E_b is insufficient. However, taking the fiber E_b and remembering that it is acted on, in a homotopy sense, by the the loop space $\Omega_b B$ is great—it lets you completely recover the map f over the connected component containing b. Importing this into algebraic geometry gives a way of working with the $k[\![\beta]\!]$-linear structure using algebro-geometric methods.

Each viewpoint has its pros and cons for our purposes:

– Viewpoint (ii) is well-suited for reducing questions about $\mathrm{PreMF}(M, f)$ (resp., $\mathrm{MF}(M, f)$) over $k[\![\beta]\!]$ (resp., $k(\!(\beta)\!)$) to questions about coherent complexes in (derived) algebraic geometry. This will be the focus of Section 3.1 and Section 4.1. Using this we deduce $k[\![\beta]\!]$- and $k(\!(\beta)\!)$-linear versions of the tensor product theorem (Theorem 3.2.1.3) and identifications of functor categories (Theorem 3.2.2.3). It is worth noting that in the $k[\![\beta]\!]$-linear context, certain *support conditions* appear naturally.[5]

– Viewpoint (iii) is well-suited to formulating comparisons between structures and invariants for $\mathrm{DCoh}(M)$ over k, and $\mathrm{PreMF}(M, f)$ (resp., $\mathrm{MF}(M, f)$) over $k[\![\beta]\!]$ (resp., $k(\!(\beta)\!)$). It is needed for the finer points of many of our applications, such as computing E_2-algebraic structure on the Hochschild invariants in chapter 7.

1.3 Summary of results

For us an *LG pair* (M, f) consists of a smooth orbifold M and a map $f\colon M \to \mathbb{A}^1$, not necessarily flat. Then $\mathrm{PreMF}(M, f) = \mathrm{DCoh}(M \times_{\mathbb{A}^1} 0)$ is coherent complexes on the derived fiber product, equipped with a certain $k[\![\beta]\!]$-linear structure depending on f; $\mathrm{MF}(M, f) = \mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$ is its two-periodic version. Our main results are variants of the "tensor product theorem" and description of functor categories in the $k[\![\beta]\!]$-linear context:

Theorem 3.2.1.3 ("Thom-Sebastiani"). Suppose (M, f) and (N, g) are two LG pairs. Set $M_0 = f^{-1}(0)$, $N_0 = g^{-1}(0)$, $(M \times N)_0 = (f \boxplus g)^{-1}(0)$, and let $\ell : M_0 \times N_0 \to (M \times N)_0$ be the inclusion. Then, there is a $k[\![\beta]\!]$-linear equivalence

$$\ell_*(- \boxtimes -)\colon \mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} \mathrm{PreMF}(N, g) \xrightarrow{\sim} \mathrm{PreMF}_{M_0 \times N_0}(M \times N, f \boxplus g)$$

[5] In the 2-periodic case, it is largely possible to ignore these by e.g., summing over critical values.↑

Theorem 3.2.2.2 and Theorem 3.2.2.3 ("Duality and Functors"). Let (M, f), (N, g), etc. be as before. Grothendieck duality for $\mathrm{DCoh}(M_0)$ lifts to a $k[\![\beta]\!]$-linear anti-equivalence $\mathrm{PreMF}(M, f)^{\mathrm{op}} \simeq \mathrm{PreMF}(M, -f)$, and with the above this induces a $k[\![\beta]\!]$-linear equivalence of dg-categories

$$\begin{aligned}\mathrm{Fun}^L_{k[\![\beta]\!]}\left(\mathrm{PreMF}^\infty(M, f), \mathrm{PreMF}^\infty(N, g)\right) &= \mathrm{PreMF}^\infty(M, -f)\widehat{\otimes}_{k[\![\beta]\!]}\mathrm{PreMF}^\infty(N, g)\\ &= \mathrm{PreMF}^\infty_{M_0\times N_0}(M\times N, -f \boxplus g)\end{aligned}$$

In case $(M, f) = (N, g)$, there are explicit descriptions of the identity functor and "evaluation"(=Hochschild homology).

The reader is directed to the actual statements of the Theorems below for variants: support conditions, the 2-periodic versions, and removing support conditions in the 2-periodic setting.

As applications of the main results, we establish several expected computations and properties MF.

Theorem 6.1.1.1. Suppose (M, f) is an LG pair. Then, $\mathrm{MF}(M, f)$ is smooth over $k(\!(\beta)\!)$, and is proper over $k(\!(\beta)\!)$ provided that $\mathrm{crit}(f) \cap f^{-1}(0)$ is proper.

Theorem 6.1.3.4. Suppose (M, f) is an LG pair, $m = \dim M$, and that M is equipped with a volume form $\mathrm{vol}_M : \mathcal{O}_m \simeq \omega_M[-m]\,(= \Omega^m_M)$. Then, vol_M determines an m-Calabi-Yau structure (in the smooth, non-proper sense) on $\mathrm{MF}(M, f)$ over $k(\!(\beta)\!)$.

In the case of computing Hochschild invariants, we also obtain $k[\![\beta]\!]$-linear refinements.

Theorem 6.1.2.5. The expected computations of 2-periodic Hochschild invariants for matrix factorizations hold: There is a homotopy S^1-action on $\mathbf{HH}_\bullet(M)$ and $\mathbf{HH}^\bullet(M)$, whose B-operator can be identified under HKR with $df \wedge -$ and i_{df}, such that

$$\mathbf{HH}_\bullet^{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}(M, f)\right) = \left(\mathbf{HH}^k_\bullet(M)\right)^{\mathrm{Tate}} \left(\simeq \mathrm{R}\Gamma\left(M, \Omega^\bullet_M(\!(\beta)\!), \beta\cdot(-df \wedge -)\right)\right)$$

$$\mathbf{HH}^\bullet_{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}(M, f)\right) = \left(\mathbf{HH}^\bullet_k(M)\right)^{\mathrm{Tate}} \left(\simeq \mathrm{R}\Gamma\left(M, T^\bullet_M(\!(\beta)\!), \beta\cdot i_{df}\right)\right)$$

Moreover, there are $k[\![\beta]\!]$-linear refinements, which in the case of M a scheme can be explicitly identified via HKR and local cohomology

$$\mathbf{HH}_\bullet^{k[\![\beta]\!]}\left(\mathrm{PreMF}(M, f)\right) = \mathbf{HH}^k_\bullet(\mathrm{DCoh}_{M_0}(M))^{S^1} \left(\simeq \mathrm{R}\Gamma_{M_0}\left([\Omega^\bullet_M[\![\beta]\!], \beta\cdot(-df \wedge -)]\right)\right)$$

(The reader is directed to the body of the text for a more precise statement.)

Furthermore, the above Theorem gets spruced up:

- In chapter 5, we see that the S^1-action on $\mathbf{HH}_\bullet(M)$ and $\mathbf{HH}^\bullet(M)$ comes from a $B\widehat{\mathbb{G}_a}$-action on the dg-category $\mathrm{DCoh}(M)$ itself. Consequently, the description as Tate-construction is compatible with all the functorially attached structures on Hochschild invariants ($\mathrm{SO}(2)$-action on $\mathbf{HH}_\bullet$, E_2-algebra structure on $\mathbf{HH}^\bullet$, etc.).
- In chapter 7, we get a handle on this S^1-action in terms of the "adjoint action" by $f \in \mathbf{HH}^0(M)$. Leverage a formality theorem of Dolgushev-Tamarkin-Tsygan [DTT], we show that the above descriptions in terms of differential forms and polyvector fields can be made to respect the "homotopy Calculus" structure on both sides.

Along the way, we stop for a few other nearby applications: Using a mild extension of the "tensor product theorem" we prove an extension of Knörrer periodicity allowing one to discard metabolic quadratic bundles; motivated by this, we identify matrix factorizations for quadratic bundles with sheaves over Clifford algebras (and the $k[\![\beta]\!]$-linear analog, upon imposing a support condition).

Theorem 6.2.1.7 and Theorem 6.2.3.4. Suppose (M, f) is an LG pair, and $\mathcal{Q}$ a non-degenerate quadratic bundle over M. View $(\mathcal{Q}, q)$ as an LG pair. Then, the structure sheaf $\mathcal{O}_M$ induces equivalences

$$\mathrm{PreMF}^{\infty}_{M}(\mathcal{Q}, q) \simeq \mathrm{PreCliff}_{\mathcal{O}_M}(\mathcal{Q})\text{-mod}(\mathrm{QC}(M)) \text{ and } \mathrm{MF}^{\infty}(\mathcal{Q}, q) \simeq \mathrm{Cliff}_{\mathcal{O}_M}(\mathcal{Q})_{\mathbb{Z}/2}\text{-mod}_{dg\mathbb{Z}/2}(\mathrm{QC}(M))$$

Exterior product over M induces an equivalence

$$\mathrm{PreMF}(M, f) \otimes_{\mathrm{Perf}(M)[\![\beta]\!]} \mathrm{PreMF}_M(\mathcal{Q}, q) \xrightarrow{\sim} \mathrm{PreMF}(\mathcal{Q}, f + q)$$

and its 2-periodic analog. Finally, if $\mathcal{Q}$ is *metabolic*, in the sense of admitting a Lagrangian sub-bundle $\mathcal{L} \subset \mathcal{Q}$, then tensoring by $\mathcal{O}_{\mathcal{L}}$ induces an equivalence

$$\mathcal{O}_{\mathcal{L}} \otimes - \colon \mathrm{Perf}(M)[\![\beta]\!] = \mathrm{MF}(M, 0) \longrightarrow \mathrm{MF}(\mathcal{Q}, q).$$

1.3.1 Comments

A few comments on the main ingredients and tools:

(i) In addition to the language of derived algebraic geometry, we make use of Grothendieck duality/the upper-shriek functor for $\mathrm{QC}^!$ of derived schemes (and certain nice derived DM stacks). Since the first version of this document became available, a preliminary reference for this has appeared in [G]. Nevertheless, we have chosen to present (in Section 3.2) a proof of the Main Theorems which we hope is reasonably concrete and minimizes the use of this general machinery. If one is only interested in matrix factorizations for a flat map $f\colon M \to \mathbb{A}^1$ from a smooth scheme, one only needs extra input in one place: determining the kernel of the identity functor in Theorem 3.2.1.3 uses duality and base-change properties on some very mild derived schemes. Nevertheless, even this proof uses some results from Section A.2—the Appendix contains some fairly meaty arguments.

(ii) We were heavily inspired by Constantin Teleman's description of PreMF (resp., MF) arising as S^1-invariants (resp., Tate construction) of perfect complexes on the total space: this proves to be a great organizational principle, as well as a useful tool for obtaining natural comparison maps.

(iii) The $k[\![\beta]\!]$-linear structure, and its relation to DSing of a hypersurface, is well-known in the commutative-algebra literature (as "cohomology operations" on DCoh of ci rings). Seidel's preprint [S] also explicitly mentions the description of DSing as arising by inverting β on PreMF.

(iv) We freely use abstract ∞-categorical tools from [L9] to make life easier: relative tensor products, Ind completions, limits and colimits (esp. in Pr^L and Pr^R). Similarly, we use results in "higher algebra" from [L6]: structured module categories for E_k-algebras, etc.

Chapter 2

Notation and background

2.1 Gradings, categories, etc.

2.1.1 Grading conventions

- We work throughout over a fixed characteristic zero field k.
- We use *homological grading* conventions (i.e., differentials increase degree) and we write π_{-i} for $H^i = H_{-i}$; e.g., $\mathrm{Ext}^i(M,N) = \pi_{-i}\,\mathrm{RHom}(M,N)$). For a chain complex M, the symbol $M[n]$ denotes the chain complex with $M[n]_k = M_{k-n}$ (i.e., if M is in degree 0, then $M[n]$ is in homological degree $+n$).
- $k[\![\beta]\!]$, $k(\!(\beta)\!)$ will denote the graded-commutative k-algebras with (homological) $\deg\beta = -2$. Fix once and for all an equivalence $C^*(BS^1;k) = k[\![\beta]\!]$ (say by $\beta \mapsto c_1(\mathcal{O}(1))$ in the Chern-Weil model for c_1).
- We will write *t-bounded-below* for what might otherwise be called *homologically bounded-below* = *cohomologically bounded-above* = *almost connective* = *right-bounded*. Similarly for *t-bounded-above* = *homologically bounded* = *truncated* = *left-bounded*; and for *t-bounded* = *(co/)homologically bounded*. For example, if A, B are discrete R-modules, then $A \overset{L}{\otimes} B$ is t-bounded-below, while $\mathrm{RHom}(A,B)$ is t-bounded-above.

2.1.2 Reminder on dg-categories and ∞-categories

For background on ∞-categories and dg-categories, the reader is direct to e.g., [L9] and [T2].

- Let $\mathbf{dgcat}_k$ be the ∞-category of k-linear dg-categories with quasi-equivalences inverted; a Theorem of Toën identifies this with (the coherent nerve of) the simplicial category whose morphisms are (Kan replacements of the nerve of) a certain full subspace of the ∞-groupoid of bimodules. Let $\mathbf{dgcat}_k^{\mathrm{idm}}$ be the ∞-categorical "Morita localization" of $\mathbf{dgcat}_k$; it may be identified with the ∞-category of *small stable idempotent complete k-linear ∞-categories* (with exact k-linear functors). Let $\mathbf{dgcat}_k^{\infty}$ denote the ∞-category of *stable cocomplete k-linear ∞-categories* (with colimit preserving k-linear functors).
- We will generally write Map for simplicial mapping spaces and RHom (with various decorations) for k-linear mapping complexes, so that e.g., $\mathrm{Map}(x,y) \simeq \Omega^\infty\,\mathrm{RHom}(x,y)$. ($\Omega^\infty$ denotes taking the infinite loop space corresponding to a spectrum; if $\mathrm{RHom}(x,y)$

is viewed as a k-linear chain complex, this may be interpreted as applying the Dold-Kan construction to the connected cover $\tau_{\geq 0}\,\mathrm{RHom}(x, y)$.)

– $\mathbf{dgcat}_k^{\mathrm{idm}}$ (resp., $\mathbf{dgcat}_k^{\infty}$) is equipped with a symmetric-monoidal *tensor product* $\otimes = \otimes_k$ (resp., $\widehat{\otimes} = \widehat{\otimes}_k$). These satisfy the compatibility $\mathrm{Ind}(\mathcal{C} \otimes \mathcal{C}') = \mathrm{Ind}\,\mathcal{C} \widehat{\otimes} \mathrm{Ind}\,\mathcal{C}'$. (In particular, $\widehat{\otimes}$ preserves the property of being compactly-generated.)[1]

– Many of our dg-categories will be $R = k[\![\beta]\!]$- or $R = k(\!(\beta)\!)$-linear, in the sense of being module-categories for the symmetric-monoidal ∞-categories $\mathrm{Perf}(R) \in \mathbf{CAlg}(\mathbf{dgcat}_k^{\mathrm{idm}})$ (resp., $R\text{-mod} \in \mathbf{CAlg}(\mathbf{dgcat}_k^{\infty})$): Heuristically, this is a $\mathcal{C} \in \mathbf{dgcat}_k^{\mathrm{idm}}$ equipped with a k-linear $\otimes\colon \mathrm{Perf}(R) \times \mathcal{C} \to \mathcal{C}$ suitably compatible with $\otimes_R$ on $\mathrm{Perf}(R)$. This notion gives rise to the same ∞-categories $\mathbf{dgcat}_R^{\mathrm{idm}}$ (resp., $\mathbf{dgcat}_R^{\infty}$) as the more rigid notion of literal R-linear dg-category, but is more convenient for our purposes. If R is a commutative dga, when no confusing arises we will sometimes write R in place of $\mathrm{Perf}\,R$ or R-mod: i.e.,

$$\mathcal{C} \otimes_R \mathcal{C}' \stackrel{\mathrm{def}}{=} \mathcal{C} \otimes_{\mathrm{Perf}\,R} \mathcal{C}' \qquad \mathcal{C} \otimes_R R' \stackrel{\mathrm{def}}{=} \mathcal{C} \otimes_{\mathrm{Perf}\,R} \mathrm{Perf}\,R'$$

$$\mathcal{D} \widehat{\otimes}_R \mathcal{D}' \stackrel{\mathrm{def}}{=} \mathcal{D} \widehat{\otimes}_{R\text{-mod}} \mathcal{D}' \qquad \mathcal{D} \widehat{\otimes}_R R' \stackrel{\mathrm{def}}{=} \mathcal{D} \widehat{\otimes}_{R\text{-mod}} R'\text{-mod}$$

– The internal-Hom associated to $\widehat{\otimes}_R$ will denoted $\mathrm{Fun}_R^L(-,-)$ (the "L" standing for left-adjoint, i.e., colimit preserving); $\mathrm{Fun}_R^L(-,-)$ is explicitly a dg-category of bimodules. Similarly, $\mathrm{Fun}_R^{ex}(-,-)$ will denote the ∞-category of exact (i.e., finite limit- and colimit-preserving) functors, etc.

– If $\mathcal{C} \in \mathbf{dgcat}_R^{\mathrm{idm}}$ or $\mathbf{dgcat}_R^{\infty}$, then there is a functor $\mathrm{RHom}_{\mathcal{C}}^{\otimes_R}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to R\text{-mod}$ determined up to contractible choices by

$$\mathrm{Map}_{R\text{-mod}}\left(V, \mathrm{RHom}_{\mathcal{C}}^{\otimes_R}(\mathscr{F}, \mathscr{G})\right) = \mathrm{Map}_{\mathcal{C}}\left(V \otimes_R \mathscr{F}, \mathscr{G}\right) \quad \text{for} \quad V \in \mathrm{Perf}(R), \text{and } \mathscr{F}, \mathscr{G} \in \mathcal{C}$$

Similarly if R-mod is replaced by another rigid cocomplete symmetric-monoidal ∞-category (e.g., $\mathrm{QC}(X)$ for X a perfect stack).

– Pr^L (resp., Pr^R) denotes the ∞-category of *presentable ∞-categories* and left (resp., right) adjoint functors. They are anti-equivalent, admit small limits and colimits, and forgetting down to $\mathbf{Cat}_\infty$ preserves limits. Colimits in Pr^L of a diagram of compactly-generated categories along functors preserving compact objects can be computed by taking Ind of the colimit of the resulting diagram of categories of compact objects.

2.1.3 Derived schemes, stacks, etc.

Mild derived schemes will come up naturally for us. In order to be able to uniformly discuss the orbifold, and graded, contexts we will also need some mild derived stacks. The very simplest variants suffice for our desired applications, since for us all the derivedness will be affine over an underived base. Nevertheless, we find it convenient to use the general language (and in Section 4.1 and the Appendices we prove things about derived stacks more general than necessary for our applications). Our primary references for derived algebraic geometry are [L2], the DAGs, and Toën/Toën-Vezzosi. Since there does not seem to be a good universal source for notation or terminology, we make clear our choices:

[1] Warning: This use of the symbol $\widehat{\otimes}$ is *not* the same as in [L4].↑

– Our *derived rings*, $\mathbf{DRng}_k$, will be connective commutative dg-k-algebras. We say that $A \in \mathbf{DRng}_k$ is *coherent* (resp., *Noetherian*) if $\pi_0 A$ is coherent (resp., Noetherian) and each $\pi_i A$ is finitely-presented over $\pi_0 A$. Meanwhile, $\mathbf{DRng}_k^{\mathrm{fp}}$ will be the full subcategory of almost finitely-presented commutative dg-k-algebras (=those which are Noetherian with $\pi_0 A$ finitely-presented over k). A(n almost finitely-presented) *derived space* is an étale sheaf in $\mathrm{Fun}(\mathbf{DRng}_k^{\mathrm{fp}}, \mathbf{Sp})$. A(n almost finitely-presented) *derived n-stack* is a derived space which admits a smooth surjection from a disjoint union of affine schemes, such that this map is a relative derived $(n-1)$-stack. (For $n = 0$, take one of affine derived schemes, (Zariski) derived schemes, or derived algebraic spaces. The first notion gives rise to "geometric n-stack," while the last gives the one most easily comparable to usual stacks.)

– A *derived scheme* is a Zariski-locally (derived-)ringed space $X = (\mathscr{X}, \mathcal{O}_X)$ which is locally equivalent as such to the Zariski spectrum $\operatorname{Spec} A$ for $A \in \mathbf{DRng}_k$. A *derived DM stack* (resp., *derived algebraic space*) is an étale-locally (derived-)ringed topos $X = (\mathscr{X}, \mathcal{O}_X)$ which is locally equivalent as such to the étale spectrum $\operatorname{Spec} A$ for $A \in \mathbf{DRng}_k$.[2]. Having said that, we will *forget it*: We will identify (almost finitely-presented) derived schemes/derived algebraic spaces/derived DM-stacks with their functors-of-points as derived spaces, and will restrict to quasi-compact ones with affine diagonal (so that *affine* derived schemes are the building blocks).

– For a derived n-stack X, there is a universal *discrete (aka "0-truncated")* derived n-stack mapping to it: $\pi_0 X = \operatorname{Spec}_X(\pi_0 \mathcal{O}_X) \to X$. Note that this morphism is affine and indeed a closed immersion. Note also that $\pi_0 X$ is in the essential image of "ordinary" Artin n-stacks (for $n = 1$, it seems justifiable to remove the quotes around ordinary!).[3]

– For a derived stack X: $\mathrm{QC}(X)$ denotes the k-linear (stable cocomplete) ∞-category of *quasi-coherent complexes* on X; it is equipped with a natural t-structure, whose heart $\mathrm{QC}(X)^\heartsuit$ is equivalent to the (ordinary) category of quasi-coherent complexes on $\pi_0 X$. $\mathrm{Perf}(X) \subset \mathrm{QC}(X)$ is the full-subcategory of *perfect complexes*; if X is a quasi-compact and (quasi-)separated derived scheme, or more generally perfect in the sense of [BZFN], then $\mathrm{QC}(X) = \operatorname{Ind} \mathrm{Perf}(X)$. $\mathrm{PsCoh}(X) \subset \mathrm{QC}(X)$ denotes the full-subcategory of *pseudo-coherent (="almost perfect") complexes*, i.e., those $\mathscr{F} \in \mathrm{QC}(X)$ that are (locally) t-bounded-below and such that $\tau_{\leq n}\mathscr{F} \in \mathrm{QC}_{\leq n}(X)$ is compact for all $n \in \mathbb{Z}$.

– We say that a derived stack X is *coherent* (resp., *Noetherian*) if it admits an fppf surjection from $\operatorname{Spec} A$ with A a coherent (resp., Noetherian) derived ring. If X is

[2]Except for algebraic spaces, these definitions are more restrictive than those in [L2], disallowing any derived-ness in the gluing process. This is rigged so that e.g., a derived DM stack will have an underlying (1-)stack.↑

[3]In particular, for $n > 1$, $\pi_0 X$ need not be equivalent to an ordinary (1-)stack. The issue is most apparent when thinking of derived (DM) stacks in terms of ∞-topoi, where the issue is analogous to the difference between an (ordinary) DM stack and a coarse moduli space. Writing $X = (\mathscr{X}, \mathcal{O}_X)$, we have $\pi_0 X = (\mathscr{X}, \pi_0 \mathcal{O}_X)$. There *is* an underlying (ordinary) DM stack $\underline{X} = (\tau_{\leq 0} X, \pi_0 \mathcal{O}_X)$, but the natural map $\mathscr{X} \to i_{\leq 0}\tau_{\leq 0}\mathscr{X}$ need not be an equivalence. The prototypical failure mode is the following: Choose $E_\bullet$ a simplicial diagram of (ordinary) stacks étale over $\underline{X}$, and let $\mathscr{X}'$ be the ∞-topos of étale sheaves of spaces on $\tau_{\leq 0} X$ *over* the geometric realization $|E_\bullet|$; then $(\mathscr{X}', \pi_0 \mathcal{O}_X|_{\mathscr{X}'})$ is a perfectly good discrete DM stack, which is not in any reasonable way a nilthickening of an ordinary DM stack.↑

coherent, then $\mathrm{PsCoh}(X)$ admits an alternate description: $\mathscr{F} \in \mathrm{PsCoh}(X)$ iff $\mathscr{F}$ is t-bounded below and $\pi_i\mathscr{F}$ is a coherent $\pi_0 X$-module for all i. Let $\mathrm{DCoh}(X) \subset \mathrm{PsCoh}(X)$ denote the full subcategory of *coherent complexes*, i.e., complexes with locally bounded, coherent (over $\pi_0 X$), cohomology sheaves. Let $\mathrm{QC}^!(X) \stackrel{\mathrm{def}}{=} \mathrm{Ind}\,\mathrm{DCoh}(X)$ denote the ∞-category of Ind objects of $\mathrm{DCoh}(X)$ ("*Ind coherent complexes*").[4] We say that X is *regular* if $\mathrm{Perf}(X) = \mathrm{DCoh}(X)$.[5]

- We say that a derived n-stack X is *bounded* if it admits a smooth surjection $U = \mathrm{Spec}\,A \to X$ which is a bounded relative $(n-1)$-stack. A 0-stack (derived scheme or algebraic space) is bounded if it is quasi-compact and quasi-separated. This is an analog of the technical condition that a scheme is quasi-compact and quasi-separated, but buys one somewhat less: One can try to compute pushforwards via a Cech complex, but this now involves a cosimplicial totalization (rather than a finite limit) and so only commutes with colimits, finite Tor-dimension base-change, etc. on *t-bounded-above* complexes.

- With all that out of the way, we now introduce two convenient conditions on a derived stack X (the conditions are somewhat redundant for clarity):

$$X \text{ is Noetherian, has affine diagonal, and is perfect} \tag{$\star$}$$

$$X \text{ is Noetherian, has finite diagonal, is perfect, and is Deligne-Mumford} \tag{$\star_{\mathrm{F}}$}$$

A $(\star)$ (resp., $(\star_{\mathrm{F}})$) morphism $f : X \to Y$ of Noetherian derived stacks is one such that $X \times_Y \mathrm{Spec}\,A$ is an $(\star)$ (resp., $(\star_{\mathrm{F}})$) derived stack for any $\mathrm{Spec}\,A \to Y$ almost of finite-presentation.

It will be our *standing assumption* that any derived stack (including plain schemes) for which we consider DCoh or $\mathrm{QC}^!$ satisfy condition $(\star)$ (and usually they will satisfy $(\star_{\mathrm{F}})$ and be almost finitely-presented over k). Note that $(\star_{\mathrm{F}})$ holds for separated Noetherian schemes, and in char. 0 for separated Noetherian DM stacks with affine diagonal whose coarse moduli space is a scheme. Both conditions pass to quotients by finite group schemes (in char. 0), and BG is $(\star)$ for G reductive (in char. 0); both pass to things quasi-projective over a base, and are stable under fiber products provided one of the maps is almost of finite-presentation (to preserves the Noetherian condition).

2.1.4 LG pairs

- An *LG pair* (M, f) is a pair consisting of a smooth $(\star_{\mathrm{F}})$ stack (*"orbifold"*) M over k, and a morphism $f\colon M \to \mathbb{A}^1$. (We do not require f to be non-zero. However, if f is not flat, various fiber products throughout the paper must be taken in the derived sense.) If (M, f), (N, g) are two LG pairs, define the *Thom-Sebastiani sum* LG pair to be $(M \times N, f \boxplus g)$ where $(f \boxplus g)(m, n) = f(m) + g(n)$.

[4] This is not the best *definition* for arbitrary X, since it does not manifestly have descent. Instead, one should make this definition on affines and then extend by gluing as in Section 4.1. But Section A.1 implies the two agree on, e.g., reasonable DM stacks.↑

[5] For $X = \mathrm{Spec}\,R$ coherent with $\pi_0 R$ a Noetherian local ring, the inclusion $\mathrm{DCoh}(X) \subset \mathrm{Perf}(X)$ is equivalent to requiring that the residue field $k = R/\mathfrak{m}$ be perfect over R. For $X = \mathrm{Spec}\,R$ coherent, the inclusion $\mathrm{Perf}(X) \subset \mathrm{DCoh}(X)$ is not automatic since R as it requires that R have only finitely many non-vanishing homotopy groups each of which is finitely-presented over $\pi_0 R$; e.g., it is satisfied for anything of finite Tor-amplitude over an underived stack. If $X = \mathrm{Spec}\,R$ coherent with $\pi_0 R$ Noetherian, it seems likely that X is regular iff $R = \pi_0 R$ is a regular ring.↑

– For an LG pair (M, f), $\mathrm{PreMF}(M, f)$ denotes the $k[\![\beta]\!]$-linear ∞-category with underlying k-linear ∞-category $\mathrm{DCoh}(M_0)$ and β acting as a "cohomological operation". See Construction 3.1.1.5 below for a geometric description of this structure. Then, $\mathrm{MF}(X, f)$ denotes the $k(\!(\beta)\!)$-linear (i.e., 2-periodic) ∞-category $\mathrm{MF}(X, f) \stackrel{\mathrm{def}}{=} \mathrm{PreMF}(X, f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$. (The relation of this to actual "matrix factorizations" is given by Prop. 3.1.4.1 and Orlov's Theorem [O2].) We also define Ind-completed versions:

$$\mathrm{PreMF}^{\infty}(X, f) \stackrel{\mathrm{def}}{=} \mathrm{Ind}\,\mathrm{PreMF}(X, f)$$

$$\mathrm{MF}^{\infty}(X, f) \stackrel{\mathrm{def}}{=} \mathrm{Ind}\,\mathrm{MF}(X, f) = \mathrm{PreMF}^{\infty}(X, f)\widehat{\otimes}_{k[\![\beta]\!]} k(\!(\beta)\!)$$

– $\Omega_M^{-\bullet} \stackrel{\mathrm{def}}{=} \oplus_i \Omega_M^i[i]$, i.e., it is placed in homologically positive degrees. Meanwhile, ω_M denotes the dualizing complex in its natural degree (not generally zero). With these conventions if M is smooth of dimension m, then $\omega_M \simeq \Omega_M^m[m]$ is in homological degree m and there is a sheaf perfect-pairing $\wedge\colon \Omega_M^{-\bullet} \otimes_{\mathcal{O}_M} \Omega_M^{-\bullet} \to \omega_M$. Similarly $T_M^{\bullet} \stackrel{\mathrm{def}}{=} \oplus_i \bigwedge^i T_M[-i]$, i.e., it is placed in homologically negative degrees.

2.2 Primer on $\mathrm{QC}^! = \mathrm{Ind}\,\mathrm{DCoh}$

2.2.0.1. The usual construction of $\mathrm{Ind}\,\mathcal{C}$, as the full subcategory of the functor category $\mathrm{Fun}(\mathcal{C}, \mathbf{Sp})$ generated under filtered colimits by the image of the Yoneda functor, provides a description

$$\mathrm{QC}^!(X) = \mathrm{Fun}^{Lex}(\mathrm{DCoh}(X)^{\mathrm{op}}, \mathbf{Sp}) \qquad \mathrm{DCoh}(X) \ni \mathscr{K} \mapsto \mathrm{RHom}(-, \mathscr{K}) \in \mathrm{Fun}^{Lex}(\mathrm{DCoh}(X)^{\mathrm{op}}, \mathbf{Sp})$$

where Fun^{Lex} denotes the full-subcategory of functors preserving finite limits. In dg-language, this translates to an identification of $\mathrm{QC}^!(X)$ with (a full subcategory of) $\mathrm{dgmod}_k(\mathrm{DCoh}(X)^{\mathrm{op}})$: (the derived category of) dg-modules over a dg-category model $\mathrm{DCoh}(X)^{\mathrm{op}}$. Our first step will be giving a slightly smaller model:

Lemma 2.2.0.2. *Suppose that X is a coherent derived stack.*

(i) Let $i\colon (\pi_0 X) \to X$ be the universal map from a discrete stack (i.e., $\mathrm{Spec}_X(\pi_0 \mathcal{O}_X) \to X$), and $i_*\colon \mathrm{DCoh}((\pi_0 X)) \to \mathrm{DCoh}(X)$ the pushforward. Then, the image of i_* triangulated-generates $\mathrm{DCoh}(X)$. In fact, objects of the form $i_*\mathscr{F}$, for $\mathscr{F} \in \mathrm{DCoh}(\pi_0 X)^{\heartsuit} = \mathrm{Coh}(\pi_0 X)$, triangulated-generate $\mathrm{DCoh}(X)$.

(ii) The right-adjoint $i^!\colon \mathrm{Ind}\,\mathrm{DCoh}\,X \to \mathrm{Ind}\,\mathrm{DCoh}(\pi_0 X)$ to $i_*\colon \mathrm{Ind}\,\mathrm{DCoh}(\pi_0 X) \to \mathrm{Ind}\,\mathrm{DCoh}\,X$ is conservative.

(iii) Suppose that $\mathscr{N} \subset \pi_0 \mathcal{O}_X$ is a nilpotent ideal sheaf (e.g., the nilradical on a Noetherian derived stack). Let $i_2\colon X' = \mathrm{Spec}_X\, \pi_0 \mathcal{O}_X/\mathscr{N} \to X$ be the corresponding map from the discrete derived stack $X' = \mathrm{Spec}_X\, \pi_0 \mathcal{O}_X/\mathscr{N}$. Then, the image of $(i_2)_*$ triangulated-generates $\mathrm{DCoh}(X)$. In fact, objects of the form $(i_2)_*\mathscr{F}$, for $\mathscr{F} \in \mathrm{DCoh}(X')^{\heartsuit} = \mathrm{Coh}(X')$, triangulated-generate $\mathrm{DCoh}(X)$.

Proof. (i) Suppose $\mathscr{F} \in \mathrm{DCoh}(X)$, and consider the Postnikov stage

$$\tau_{\geq(k+1)}\mathscr{F} \longrightarrow \tau_{\geq k}\mathscr{F} \longrightarrow (\pi_k \mathscr{F})[k]$$

Note that $\pi_k\mathscr{F}$ is a coherent $\pi_0\mathcal{O}_X$-module since $\mathscr{F} \in \mathrm{DCoh}(X)$, and thus is in the essential image of i_*. Since X is quasi-compact and $\mathscr{F} \in \mathrm{DCoh}(X)$, only finitely many k are non-zero, completing the proof.

(ii) Suppose $\mathscr{F} = \text{“}\varinjlim_\alpha\text{”}\mathscr{F}_\alpha \in \operatorname{Ind}\mathrm{DCoh}\, X$ is such that $i^!\mathscr{F} = 0$. It suffices to show that $0 = \mathrm{Map}_{\operatorname{Ind}\mathrm{DCoh}\, X}(\mathscr{K}, \mathscr{F}) = \varinjlim_\alpha \mathrm{Map}_{\mathrm{QC}(X)}(\mathscr{K}, \mathscr{F}_\alpha)$ for all $\mathscr{K} \in \mathrm{DCoh}(X)$. By (i), it suffices to note that $0 = \mathrm{Map}_{\operatorname{Ind}\mathrm{DCoh}\, X}(i_*\mathscr{K}', \mathscr{F}) = \mathrm{Map}_{\operatorname{Ind}\mathrm{DCoh}\, X}(\mathscr{K}', i^!\mathscr{F})$ for all $K' \in \mathrm{DCoh}(\pi_0 X)$.

(iii) By the above it suffices to show that the triangulated closure of the image contains $i_*\mathscr{F}$ for $\mathscr{F} \in \mathrm{Coh}(\pi_0 X)$. The filtration of $\mathscr{F}$ by powers of $\mathscr{N}$

$$\mathscr{F} \supset \mathscr{N}\mathscr{F} \supset \mathscr{N}^2\mathscr{F} \supset \cdots$$

is finite by hypothesis, and each associated graded piece is in the essential image of $(i_2)_*$. □

This yields the following comforting description of $\mathrm{QC}^!(X)$:

Corollary 2.2.0.3. *Suppose X is a Noetherian derived scheme. Let $\mathcal{C}$ denote a dg-category whose objects are ordinary coherent sheaves on $\pi_0 X$ and whose morphisms are $\mathrm{RHom}_X^{\otimes_k}(i_*\mathscr{F}, i_*\mathscr{G})$. Then, $\mathrm{QC}^!(X)$ may be identified with (a full subcategory/localization of) the dg-category of dg-modules over $\mathcal{C}^{\mathrm{op}}$.*

Alternatively, let $\mathcal{C}'$ be the dg-category whose objects are coherent sheaves on $(\pi_0 X)^{\mathrm{red}}$ and whose morphisms are as above. Then, $\mathrm{QC}^!(X)$ may be identified with the (a full subcategory/localization of) dg-category of dg-modules over $(\mathcal{C}')^{\mathrm{op}}$.

2.2.0.4. In case X is a (discrete) Noetherian separated scheme, there are more explicit dg-models for $\mathrm{QC}^!(X)$ and $\mathrm{QC}^!(X)^\vee = \operatorname{Ind}(\mathrm{DCoh}(X)^{\mathrm{op}})$ in the literature:

- $K(\operatorname{Inj} X)$ the “homotopy” dg-category of (unbounded) complexes of injective quasi-coherent sheaves. This description emphasizes that “the difference” between $\mathrm{QC}^!(X)$ and $\mathrm{QC}(X)$ is that the later is complete with respect to the t-structure, i.e., acyclic objects are equivalent to 0. It models $\mathrm{QC}^!(X)$ by results of Krause.
- $K_m(\operatorname{Proj} X)$ Murfet’s “mock homotopy category of projectives” (after Jørgensen and Neeman). In the affine case, one can literally take the dg-category of (unbounded) complexes of projective quasi-coherent modules, while in general one must take a certain localization of the dg-category of (unbounded) complexes of flat quasi-coherent modules. It models $\mathrm{QC}^!(X)^\vee$ by results of Neeman and Murfet.

One can give similar dg-models in the derived setting, based on Positselski’s *coderived* and *contraderived* categories of dg-modules.

Notation 2.2.0.5. Suppose S is a perfect stack, so that $\mathrm{QC}(S) = \operatorname{Ind}\operatorname{Perf}(S)$.

- If $f \colon X \to S$ is a relative derived stack, then $\mathrm{QC}(X)$ is a $\mathrm{QC}(S)$-module category (via the symmetric monoidal pullback functor). This gives rise to an inner-Hom functor $\mathcal{RH}om^{\otimes_S}_{\mathrm{QC}(X)} \colon \mathrm{QC}(X)^{\mathrm{op}} \times \mathrm{QC}(X) \to \mathrm{QC}(S)$ characterized by

$$\mathrm{Map}_{\mathrm{QC}(S)}\left(T, \mathcal{RH}om^{\otimes_S}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G})\right) = \mathrm{Map}_{\mathrm{QC}(X)}\left(f^*T \otimes_{\mathcal{O}_X} \mathscr{F}, \mathscr{G}\right)$$

for all $T \in \mathrm{Perf}(S)$, and $\mathscr{F}, \mathscr{G} \in \mathrm{QC}(X)$. If $X = S$, then we will omit the superscript $\otimes_S$. If $S = \operatorname{Spec} k$, we will write $\mathrm{RHom}_{\mathrm{QC}(X)}$.

– If $f\colon X \to S$ is an S-scheme, then $\mathrm{QC}^!(X)$ is a $\mathrm{QC}(S)$-module category. This gives rise to $\mathcal{RH}om^{\otimes_S}_{\mathrm{QC}^!(X)}\colon \mathrm{QC}^!(X)^{\mathrm{op}} \times \mathrm{QC}^!(X) \to \mathrm{QC}(S)$ characterized by

$$\mathrm{Map}_{\mathrm{QC}(S)}\left(T, \mathcal{RH}om^{\otimes_S}_{\mathrm{QC}^!(X)}(\mathscr{F}, \mathscr{G})\right) = \mathrm{Map}_{\mathrm{QC}^!(X)}\left(f^*T \otimes_{\mathcal{O}_X} \mathscr{F}, \mathscr{G}\right)$$

for all $T \in \mathrm{Perf}(S)$ and $\mathscr{F}, \mathscr{G} \in \mathrm{QC}^!(X)$. If $X = S$, then we will omit the superscript $\otimes_S$. If $S = \operatorname{Spec} k$, we will write $\mathrm{RHom}_{\mathrm{QC}^!(X)}$. If $\mathscr{F}, \mathscr{G} \in \mathrm{DCoh}(X)$, we may write $\mathcal{RH}om_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G})$ or $\mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{F}, \mathscr{G})$: Since $\mathrm{DCoh}(X) \to \mathrm{QC}(X)$ is fully-faithful, there is no ambiguity.

Note that if $\mathscr{F} \in \mathrm{Perf}(X)$ (or $\mathscr{F} \in \mathrm{DCoh}(X)$) then $f^*T \otimes_{\mathcal{O}_X} \mathscr{F}$ is compact in $\mathrm{QC}(X)$ (or $\operatorname{Ind}\mathrm{DCoh}(X)$) for all $T \in \mathrm{Perf}(S)$, so that $\mathcal{RH}om^{\otimes_S}(\mathscr{F}, -)$ preserves colimits.

Notation 2.2.0.6. There is a natural localization functor $F_M\colon \mathrm{QC}^!(X) \to \mathrm{QC}(X)$, characterized by preserving colimits and taking the compact-objects $\mathrm{DCoh}(X)$ to themselves

$$F_M\left(\text{“}\varinjlim_\alpha\text{”}\mathscr{K}_\alpha\right) = \varinjlim_\alpha \mathscr{K}_\alpha$$

It is a colimit-preserving functor between presentable ∞-categories, and so admits a right-adjoint $F_R\colon \mathrm{QC}(X) \to \mathrm{QC}^!(X)$. In terms of the identification $\mathrm{QC}^!(X) = \mathrm{Fun}^{Lex}(\mathrm{DCoh}(X)^{\mathrm{op}}, \mathbf{Sp})$, F_R is just the restriction of the Yoneda embedding; in particular, the restriction of F_R to $\mathrm{DCoh}(X)$ is the natural inclusion $\mathrm{DCoh}(X) \to \operatorname{Ind}\mathrm{DCoh}(X)$.

The names for the functors are motivated by the following: Suppose that X is perfect and that $\mathrm{Perf}(X) \subset \mathrm{DCoh}(X)$; let $F\colon \mathrm{Perf}(X) \to \mathrm{DCoh}(X)$. Denote by $F_L\colon \operatorname{Ind}\mathrm{Perf}(X) \to \operatorname{Ind}\mathrm{DCoh}(X)$ the colimit extension of F. Then, F_L is left-adjoint to F_M, while F_M is left-adjoint to F_R (so, “left”, “middle”, and “right”).[6]

2.2.0.7. Associated to a bounded morphism $f\colon X \to Y$ of derived stacks, one can attach a variety of functors. The reader is directed to [G] for more on the construction. With notation as above:

Construction 2.2.0.8. Suppose $\mathbf{F}\colon \mathrm{QC}_{<\infty}(X) \to \mathrm{QC}_{<\infty}(Y)$ is a colimit-preserving functor (on t-bounded above quasi-coherent complexes) which is *t-bounded above* in the sense that there exists a constant N such that $\mathbf{F}(\mathrm{QC}(X)_{\leq k}) \subset \mathbf{F}(\mathrm{QC}(X)_{\leq k+N})$. Then, define ${}^!\mathbf{F}\colon \mathrm{QC}^!(X) \to \mathrm{QC}^!(Y)$ as the filtered-colimit extension of the composite

$$\mathrm{DCoh}(X) \xrightarrow{\mathbf{F}} \mathrm{QC}(Y) \xrightarrow{F_R} \mathrm{QC}^!(Y)$$

Since $F_M \circ F_R = \mathrm{id}$, it follows that $F_M \circ {}^!\mathbf{F} \circ F_R = \mathbf{F}$. The importance of the t-boundedness condition is that F_R commutes with t-bounded above (but not arbitrary) colimits, so that the condition guarantees that $F_R \circ \mathbf{F} = {}^!\mathbf{F} \circ F_R$ on t-bounded above objects.

[6] Suppose that X is perfect, so that $\mathrm{QC}(X) = \operatorname{Ind}\mathrm{Perf}(X)$, but that $\mathrm{Perf}(X) \not\subset \mathrm{DCoh}(X)$ (e.g., $X = \operatorname{Spec} R$ with R having infinitely many homotopy groups). In this case, our notation is potentially confusing since F_M need not admit a left-adjoint “F_L”: Since F_M is itself a left-adjoint, any such F_L would have to preserve compact objects; in particular, the colimit extension of $F_R|_{\mathrm{Perf}(X)}\colon \mathrm{Perf}(X) \to \operatorname{Ind}\mathrm{DCoh}(X)$ cannot be the left-adjoint of F_R in this case.↑

In particular, the t-bounded-above condition guarantees that the construction is compatible with composition of functors: If $\mathbf{F}\colon \operatorname{QC}(X) \to \operatorname{QC}(Y)$ and $\mathbf{F}'\colon \operatorname{QC}(Y) \to \operatorname{QC}(Z)$ are two functors, one would very much like for the natural map ${}^{!}\mathbf{F}' \circ {}^{!}\mathbf{F} \to {}^{!}(\mathbf{F}' \circ \mathbf{F})$ to be an equivalence. Everything is colimit-preserving, so it suffices to check on the compact objects $\mathscr{K} \in \operatorname{DCoh}(X)$, which are bounded above and remain so after applying $\mathbf{F}$, so that

$$ {}^{!}\mathbf{F}' \circ {}^{!}\mathbf{F}(\mathscr{K}) = {}^{!}\mathbf{F}' \circ F_R \circ \mathbf{F}(\mathscr{K}) = F_R \circ \mathbf{F}' \circ \mathbf{F}(\mathscr{K}) = {}^{!}(\mathbf{F}' \circ \mathbf{F})(\mathscr{K}) $$

- The functor $f_*\colon \operatorname{QC}_{<\infty}(X) \to \operatorname{QC}_{<\infty}(Y)$ is colimit-preserving and t-bounded above. Therefore, it gives rise to a functor $f_*\colon \operatorname{QC}^!(X) \to \operatorname{QC}^!(Y)$ by the above procedure. If f is a bounded relative proper algebraic space,[7] then f_* preserves compact objects.
- Provided f is of finite Tor-dimension, the functor $f^*\colon \operatorname{QC}(X) \to \operatorname{QC}(Y)$ will be colimit preserving and t-bounded above (and below). In this case, it gives rise to a functor $f^*\colon \operatorname{QC}^!(Y) \to \operatorname{QC}^!(X)$ as above. Furthermore, there is an adjunction (f^*, f_*).
- The functor $f^! = \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y\colon \operatorname{QC}_{<\infty}(Y) \to \operatorname{QC}_{<\infty}(X)$ on t-bounded above complexes is colimit preserving and t-bounded above. (In case f is of finite Tor-dimension, $f^!(-) \simeq \omega_f \otimes f^*(-)$ is well-behaved with no boundedness though still preserves boundedness.) Consequently, it gives rise to a functor $f^!\colon \operatorname{QC}^!(Y) \to \operatorname{QC}^!(X)$.
- To understand $f^!$, it will suffice for our purposes to recall explicit formulae for two special cases: If f is *finite* (i.e., affine, with $f_*\mathcal{O}_X$ pseudo-coherent) then $f^!(-) = \operatorname{RHom}_X(f_*\mathcal{O}_X, -)$ equipped with the evaluation-at-one trace map $\operatorname{tr}_f\colon f_*f^! \to \operatorname{id}$. If f is *quasi-smooth* (i.e., finite-presentation and $\mathbb{L}_f$ of Tor-amplitude in $[0,1]$), then $f^!(-) = \det \mathbb{L}_f \otimes f^*(-)$ equipped with the Berezinian integration trace map $\operatorname{tr}_f\colon f_*f^! \to \operatorname{id}$.

2.2.0.9. The natural functors on QC, f^* and f_*, are simply adjoint and so determine one another. In contrast, properly spelling out the relations between the two natural functors, $f^!$ and f_*, on $\operatorname{QC}^!$ requires some $(\infty, 2)$-categorical structures which we won't get into here (e.g., one needs to remember the transformation $\operatorname{tr}_f\colon f_*f^! \to \operatorname{id}$ when it exists, etc.). Instead, we'll just mention a few facts (that hold in say, the ($\star_{\mathrm{F}}$) case)

The formation of f_* commutes with flat base-change on the target. The formation of $f^!$ commutes with flat base-change on the target and étale base-change on the source. If f is finite[8] the natural transformation $\operatorname{tr}_f\colon f_*f^! \to \operatorname{id}$ is the co-unit of an adjunction $(f_*, f^!)$.

Given a commutative square

$$ \begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array} $$

there is an equivalence $(g')_*(f')^! \simeq f^! g_*$, e.g., in case f proper as the composite

$$ (g')_*(f')^! \xrightarrow{\operatorname{cotr}_f((g')_*(f')^!)} f^! f_*(g')_*(f')^! \xrightarrow{\sim} f^! g_*(f')_*(f')^! \xrightarrow{f^! g_*(\operatorname{tr}_{f'})} f^! g_* $$

and in case f smooth a map the other way deduced from the projection formula, base-change, and the map $(g')^* \det \mathbb{L}_f \to \det \mathbb{L}_{f'}$. This natural transformation is an equivalence when

[7] Or e.g., a sufficiently nice bounded relative proper DM stack in characteristic zero.↑

[8] or with more difficulty: a bounded relative proper algebraic space, or a sufficiently nice bounded relative proper DM stack in characteristic zero↑

the square is Cartesian: Using compatibilities with base-change, the claim is étale local on X and Y, so we reduce to the case where $f\colon X \to Y$ admits a factorization as a finite morphism followed by a smooth morphism; it then remains to check (using the standard QC tools, e.g., base-change for star pullback, the projection formula, etc.) that the natural transformation is an equivalence in each of the two cases.

Chapter 3

From coherent complexes to matrix factorizations: Derived (based) loop spaces, and first proofs of Theorems

3.1 Generalities on PreMF and MF

3.1.1 Preliminaries

Construction 3.1.1.1. For the duration of this section, set

$$\mathbb{B} = 0 \times_{\mathbb{A}^1} 0 = \operatorname{Spec} k[B]/B^2 \qquad \deg B = +1$$

$\mathbb{B}$ admits the structure of *derived group scheme* (i.e., its functor of points admits a factorization $\mathbf{DRng} \to \mathbf{Mon}^{gp}(\mathbf{sSet}) \to \mathbf{Kan}$, where $\mathbf{Mon}^{gp}(\mathbf{sSet})$ denotes group-like Segal-style monoids in $\mathbf{sSet}$) by "composition of loops". Since $\mathbb{A}^1 = \mathbb{G}_a$ is a commutative group scheme, $\mathbb{B}$ admits the structure of *commutative derived group scheme* (i.e., its functor of points admits a factorization $\mathbf{DRng} \to \mathbf{sAb} \to \mathbf{sSet}$) by considering "pointwise addition of loops." These two structures are in fact strictly compatible (i.e., determine a factorization of the functor of points as $\mathbf{DRng} \to \mathbf{Mon}^{gp}(\mathbf{sAb}) \to \mathbf{sSet}$ refining each of the other two in the obvious way).

– The ("loop composition") product $\mu\colon \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ and identity $\mathrm{id}\colon \mathrm{pt} \to \mathbb{B}$ may be explicitly identified as

$$\mu\colon \mathbb{B} \times \mathbb{B} = (0 \times_{\mathbb{A}^1} 0) \times (0 \times_{\mathbb{A}^1} 0) \simeq 0 \times_{\mathbb{A}^1} 0 \times_{\mathbb{A}^1} 0 \xrightarrow{p_{13}} 0 \times_{\mathbb{A}^1} 0$$

$$\mathrm{id}\colon \mathrm{pt} \xrightarrow{\Delta} \mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt} = \mathbb{B}$$

The rest of the Segal-monoid structure admits a similar description via projections and diagonals. A homotopy inverse is given by the explicit anti-isomorphism $i\colon \mathbb{B} \simeq \mathbb{B}^{\mathrm{op}}$ which on underlying space can be identified with

$$i\colon \mathbb{B} = \mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt} \xrightarrow{\text{switch}} \mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt} = \mathbb{B}$$

– The ("pointwise addition") product $+\colon \mathbb{B}\times\mathbb{B}\to\mathbb{B}$, identity $0\colon \mathrm{pt}\to\mathbb{B}$, and inverse $-\colon \mathbb{B}\simeq\mathbb{B}^{\mathrm{op}}$ may be explicitly identified as follows: The commutative diagram

$$\begin{array}{ccc} 0\times 0 & \xrightarrow{=} & 0 \\ \downarrow & & \downarrow \\ \mathbb{A}^1\times\mathbb{A}^1 & \xrightarrow{+} & \mathbb{A}^1 \end{array}$$

gives rise to a map

$$+\colon \mathbb{B}\times\mathbb{B}\simeq(\mathrm{pt}\times\mathrm{pt})\times_{\mathbb{A}^1\times\mathbb{A}^1}\mathrm{pt}\times\mathrm{pt}\longrightarrow \mathrm{pt}\times^1_{\mathbb{A}}\mathrm{pt}=\mathbb{B}$$

Analogously, base changing the identity $0\to\mathbb{A}^1$ and the inverse map $-\colon \mathbb{A}^1\to\mathbb{A}^1$ one obtains maps

$$0\colon \mathrm{pt}=0\times_0 0\longrightarrow 0\times_{\mathbb{A}^1}0=\mathbb{B}$$

$$-\colon \mathbb{B}=0\times_{\mathbb{A}^1}0\longrightarrow 0\times_{\mathbb{A}^1}0=\mathbb{B}$$

and an (anti-)isomorphism $-\colon \mathbb{B}\simeq\mathbb{B}^{\mathrm{op}}$.

– For $R\in\mathbf{DRng}$, let $R_\bullet=\mathrm{Map}_{\mathbf{DRng}}(k[x],R)\in\mathbf{sAb}$. In terms of functor of points on $\mathbf{DRng}$, we have

$$\mathbb{B}(R_\bullet)=\mathrm{Map}_*(S^1,R_\bullet)$$

which is equipped with a Segal-monoid structure ("loop composition") via mapping in wedges of length n

$$[n]\mapsto \mathrm{Map}_*(\sqcup_n\Delta^1/\sqcup_n\Delta^0,R_\bullet)$$

For any pointed simplicial set X, $\mathrm{Map}_*(X,R_\bullet)$ is naturally a simplicial abelian group via the composite

$$\mathrm{Map}_*(X,R_\bullet)^2=\mathrm{Map}_*(X,R_\bullet^2)\to\mathrm{Map}_*(X,R_\bullet)$$

providing the lift to $\mathbf{Mon}^{gp}(\mathbf{sAb})$.

We will heuristically write these (indicating e.g., maps on (π_0 of) functor of points) as

$$\mu\colon \mathbb{B}(R_\bullet)\times\mathbb{B}(R_\bullet)\ni([h_1\colon 0\to 0],[h_2\colon 0\to 0])\mapsto\Big([h_1\cdot h_2\colon 0\overset{h_1}{\to}0\overset{h_2}{\to}0]\Big)\in\mathbb{B}(R_\bullet)$$

and

$$+\colon \mathbb{B}(R_\bullet)\times\mathbb{B}(R_\bullet)\ni([h_1\colon 0\to 0],[h_2\colon 0\to 0])\mapsto([h_1+h_2\colon 0\to 0])\in\mathbb{B}(R_\bullet)$$

Construction 3.1.1.2. Let $(\mathrm{QC}^!(\mathbb{B}),\circ)^\otimes$ denote $\mathrm{QC}^!(\mathbb{B})$ equipped with its symmetric monoidal convolution product: $\mathscr{G}\circ\mathscr{F}=+_*(\mathscr{G}\boxtimes\mathscr{F})$.

More precisely: Construction 3.1.1.1 provides a lift of $\mathbb{B}$ to $(\mathbb{B},\mu,+)\in\mathbf{Mon}(\mathbf{CMon}(\text{Der. Sch.}))$.

Composing with the lax symmetric monoidal (via exterior product) functor[1]

$$X \mapsto \mathrm{QC}^!(X), \qquad f \mapsto f_*$$

one obtains $(\mathrm{QC}^!(\mathbb{B}), \circ_\mu, \circ_+) \in \mathbf{Alg}(\mathbf{CAlg}(\mathbf{dgcat}_k^\infty))$. Finally, $(\mathrm{QC}^!(\mathbb{B}), \circ)^\otimes$ is the image of this under the forgetful functor to $\mathbf{CAlg}(\mathbf{dgcat}_k)$.

Remark 3.1.1.3. One can give explicit dg-algebra models for the two products on $\mathbb{B}$, for the two actions, and an equivalence between the two (as well as an equivalence with a convenient smaller non-Segal model for loop composition).

Consider the following diagram of cosimplicial commutative dg-algebras:

$$\left\{ k\,\overbrace{[B_1, \dots, B_\bullet]}^{\deg B_i = +1} \right\} \longrightarrow \left\{ k\,[x_1, \dots, x_\bullet]\,\overbrace{\begin{bmatrix} \epsilon_1^1, \dots, \epsilon_1^\bullet \\ \epsilon_2^1, \dots, \epsilon_2^\bullet \end{bmatrix}}^{\deg \epsilon_i^j = +1} \Big/ \left(d\epsilon_i^j = x_i \right) \right\} \longrightarrow \left\{ k[x]\,\overbrace{[\gamma_1, \dots, \gamma_{\bullet+1}]}^{\deg \gamma_i = +1} / (d\gamma_i = x) \right\}$$

where the cosimplicial structure maps, and morphisms, are

– The middle term models the co-commutative "pointwise-addition" co-multiplication (and co-identity) on $k[x][\epsilon_1, \epsilon_2]$

$$\Delta_+(x) = x \otimes 1 + 1 \otimes x \qquad \Delta_+(\epsilon_1) = \epsilon_1 \otimes 1 + 1 \otimes \epsilon_1 \qquad \mathrm{coid}_+(x) = 0$$

under the evident identification of the n-th term with the n-fold tensor product of $k[x][\epsilon_1, \epsilon_2]$.

– The right-hand term models the (Segal-style) "loop composition" co-multiplication (and co-identity) on $k[x][\gamma_1, \gamma_2]$ (note that the n-th term is quasi-isomorphic to the n-fold tensor product, though isn't strictly isomorphic to it).

– The left-hand term gives a compact model for both. It comes from the co-commutative co-multiplication on $k[B]/B^2$ given by $\Delta(B) = B \otimes 1 + 1 \otimes B$, $\mathrm{coid}(B) = 0$.

– The right-hand map sends all x_i to x, and $\epsilon_j^i \mapsto \gamma_{i+j-1}$. It is a weak equivalence.

– The left-hand maps sends $B_i \mapsto \epsilon_1^i - \epsilon_2^i$. It is a weak equivalence.

The next (standard) Proposition is the starting point for this Section. Morally, it is the following Koszul duality computation: One identifies $\mathcal{O}_\mathbb{B} \simeq C_*(S^1; k)$ as E_∞-coalgebra in dg-algebras, so that a cobar construction yields $k[\![\beta]\!] \simeq C^*(BS^1; k)$ as E_∞-algebra. (Alternatively, replace S^1 by the abelian dg-Lie algebra $k[+1]$: $U(k[+1]) \simeq \mathcal{O}_\mathbb{B}$ and $C^*(k[+1]) = k[\![\beta]\!]$.)

Proposition 3.1.1.4. *There is a symmetric monoidal equivalence* $(\mathrm{QC}^!(\mathbb{B}), \circ)^\otimes \simeq (k[\![\beta]\!]\text{-mod}, \otimes_{k[\![\beta]\!]})^\otimes$, *given on compact objects by (a suitable enrichment of)*

$$\mathrm{DCoh}(\mathbb{B}) \ni V \mapsto V^{S^1} = \mathrm{RHom}_\mathbb{B}(k, V) \in \mathrm{Perf}\, k[\![\beta]\!]$$

[1] In general, the correct way to say this would require considering a suitable coCartesian fibration $(\widetilde{\mathrm{QC}^!}, f_*) \to \mathrm{Der.\ Sch}$, and then pulling back along the $\Delta^{\mathrm{op}} \times \Gamma$-shaped diagram encoding $(\mathbb{B}, \mu, +)$. In the present case, however, all the maps are finite so that it is not hard to give a strictly functorial diagram of categories.↑

Proof. It suffices to prove the equivalence on compact objects. We will carry out the computation in the explicit (characteristic zero) dg-model for $\mathcal{O}_{\mathbb{B}}$ of Remark 3.1.1.3. Let $\mathrm{Cpx}(\mathbb{B})$ denote the (ordinary) category of dg-$\mathcal{O}_{\mathbb{B}}$-modules, and $\mathrm{Cpx}(k[\![\beta]\!])$ the (ordinary) category of dg-$k[\![\beta]\!]$-modules.

Identify

$$\mathcal{O}_{\mathbb{B}} = k[x]\ \underbrace{[\epsilon_1, \epsilon_2]}_{\deg \epsilon_i = +1}\ / \begin{pmatrix} \epsilon_i^2 = 0 \\ d\epsilon_i = x \end{pmatrix}$$

as commutative dg-k-algebra (recall, related to the smaller model by $B = \epsilon_1 - \epsilon_2$). The $+$-comultiplication, -coidentity, and -coinverse of Remark 3.1.1.3 make $\mathcal{O}_{\mathbb{B}}$ into a cocommutative, commutative, dg-Hopf algebra. Then, Δ_+, coid_+, and $- \otimes_k -$ equip the (ordinary) category $\mathrm{Cpx}(\mathbb{B})$ with a symmetric monoidal structure by setting

$$M \overset{\circ}{\otimes} M' \overset{\text{def}}{=} (\Delta_+)_*(M \boxtimes_k M')$$

with unit $k = \mathrm{coid}_+(k)$ and the evident associativity, unitality, and commutativity constraints coming from those for $\otimes_k$ on complexes of k-modules.

Recall the Koszul-Tate semi-free resolution

$$k \sim \underbrace{\mathcal{O}_{\mathbb{B}}\, [u^m/m!]}_{\deg u = +2} / \{du = \epsilon_1 - \epsilon_2\}$$

on which $k[\![\beta]\!]$ acts by $\beta = d/du$. This gives rise to the usual explicit model for $(-)^{S^1}$ (recall $B = \epsilon_1 - \epsilon_2$) as a functor on (ordinary) categories of complexes

$$((V, d_{int}))^{S^1} = \mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}\left(\mathcal{O}_{\mathbb{B}}\, [u^m/m!]\, , (V, d_{int})\right) = (V[\![\beta]\!], d_{int} + \beta B)$$

It will be more convenient for our purposes to instead work with the functor

$$\mathbf{F}\colon\ \mathrm{Cpx}(\mathbb{B}) \to \mathrm{Cpx}(k[\![\beta]\!]) \qquad \mathbf{F}(V) = V \otimes_{\mathcal{O}_{\mathbb{B}}} (\mathcal{O}_{\mathbb{B}}[\![\beta]\!], \beta B) = (V[\beta], d_{int} + \beta B)$$

Note that for V bounded-above (resp., homologically) the natural map $\mathbf{F}(V) = V \otimes_{\mathcal{O}_{\mathbb{B}}} (\mathcal{O}_{\mathbb{B}}[\![\beta]\!], \beta B) \to V^{S^1}$ is an isomorphism (resp., equivalence).[2] The functor $\mathbf{F}$ is monoidal via the natural isomorphism

$$\begin{aligned} \mathbf{F}(V, d_{int}) \otimes_{k[\![\beta]\!]} \mathbf{F}(V', d'_{int}) &= (V[\beta], d_{int} + \beta B) \otimes_{k[\![\beta]\!]} \left(V'[\beta], d'_{int} + \beta B\right) \\ &\longrightarrow \left((V \otimes_k V')[\beta], d_{int} \otimes 1 + 1 \otimes d'_{int} + \beta(B \otimes 1 + 1 \otimes B)\right) \\ &= \mathbf{F}\left[(V, d_{int}) \overset{\circ}{\otimes} (V', d'_{int})\right] \end{aligned}$$

and the equality $\mathbf{F}(k) = k[\![\beta]\!]$ of tensor units, evident compatibility with associativity, etc. The symmetry isomorphisms on both sides are given by the usual graded-commutativity

[2] On homologically bounded-above complexes, it follows that $\mathbf{F}$ preserves quasi-isomorphisms. On arbitrary complexes it need not: Say that a map $\phi : V \to V'$ is an $\mathbf{F}$-equivalence if $\mathbf{F}(\phi)$ is a quasi-isomorphism, and let $\mathbf{F}^{\sim}$ denote the collection of $\mathbf{F}$-equivalences. One can show that: every $\mathbf{F}$-equivalence is a quasi-isomorphism, but not vice versa; the localization $\mathrm{Cpx}(\mathcal{O}_{\mathbb{B}})[(\mathbf{F}^{\sim})^{-1}]$ is naturally identified with $\mathrm{Ind}\,\mathrm{DCoh}(\mathbb{B})$, so that $\mathbf{F}$ induces a functor $\mathrm{Ind}\,\mathrm{DCoh}(\mathbb{B}) \to k[\![\beta]\!]$-mod which one can show is an equivalence; since every $\mathbf{F}$-equivalence is a quasi-isomorphism, we obtain $\mathrm{Cpx}(\mathcal{O}_{\mathbb{B}})[(\mathbf{F}^{\sim})^{-1}] \to \mathrm{Cpx}(\mathcal{O}_{\mathbb{B}})[\mathrm{qiso}^{-1}] = \mathrm{QC}(\mathbb{B})$ which coincides via the above with the usual localization functor.↑

rules, and β is even, so that $\mathbf{F}$ is symmetric monoidal.

Note that the symmetric monoidal unit $\mathcal{O}_{\mathrm{id}} = k \in \mathrm{DCoh}(\mathbb{B})$ generates $\mathrm{DCoh}(\mathbb{B})$ under cones and shifts (c.f., Lemma 2.2.0.2): For any $V \in \mathrm{DCoh}(\mathbb{B})$ simply consider the finite Postnikov stages $\tau_{\geq}(m+1)V \to \tau_{\geq}mV \to (\pi_m V)[m]$, and observe that $\pi_m V[m]$ is a $\pi_0\mathbb{B} = k$-module. It follows that $\mathbf{F} \simeq (-)^{S^1}$ takes $\mathrm{DCoh}(\mathbb{B})$ to $\mathrm{Perf}\, k[\![\beta]\!]$.We claim that $\mathbf{F}$ can be used to construct a symmetric-monoidal functor of ∞-categories $(\mathrm{DCoh}(\mathbb{B}), \circ)^{\otimes} \to (\mathrm{Perf}\, k[\![\beta]\!], \otimes_{k[\![\beta]\!]})^{\otimes}$ which is equivalent to $(-)^{S^1}$ on underlying categories. Assuming the claim, we complete the proof. A symmetric-monoidal functor is an equivalence iff it is so on underlying ∞-categories so that it suffices to show that $\mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}(k, -)\colon \mathrm{DCoh}(\mathbb{B}) \to \mathrm{Perf}\, k[\![\beta]\!]$ is an equivalence. The map of complexes $k[\![\beta]\!] \to \mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}(k, k)$ described above is evidently a quasi-isomorphism. Since $\mathrm{DCoh}(\mathbb{B})$ is stable, idempotent complete, and generated (in the stable, idempotent complete sense) by k it follows by Morita theory that the functor $\mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}(k, -)$ is an equivalence.

The rest of the proof will be devoted to giving the details of obtaining from $\mathbf{F}$ a symmetric monoidal functor of ∞-categories:

– Equip $\mathrm{Cpx}(\mathcal{O}_{\mathbb{B}})$ with its *injective* model structure, i.e., the weak-equivalences and cofibrations are maps which are so on underlying complexes. Together with $\overset{\circ}{\otimes}$ above, this makes it into a simplicial symmetric-monoidal model category in the sense of [L3, Def.4.3.11]; e.g., the compatibilities of tensor and weak-equivalences/cofibrations follow from the analogous statements for chain complexes over k. It follows that the symmetric-monoidal ∞-category $(\mathrm{DCoh}(\mathbb{B}), \circ)^{\otimes} \to N(\Gamma)^{\otimes}$ admits a description as the homotopy coherent nerve of a fibrant simplicial category $(\mathrm{Cpx}^{\circ,\mathrm{DCoh}}(\mathcal{O}_{\mathbb{B}}))^{\otimes}$ over Γ formed as follows: Its objects are tuples $(\langle n\rangle, C_1, \ldots, C_n)$ with $\langle n\rangle \in \Gamma$ and with each C_i a bounded-above injective-fibrant dg-$\mathcal{O}_{\mathbb{B}}$-module with bounded coherent cohomology, and its simplicial mapping spaces are

$$\mathrm{Map}\left((\langle n\rangle, C_1 \ldots, C_n), (\langle m\rangle, C'_1, \ldots, C'_m)\right) = \coprod_{\alpha:\langle n\rangle\to\langle m\rangle} \prod_{1\leq j\leq n} \mathrm{Map}_{\mathrm{Cpx}(\mathcal{O}_{\mathbb{B}})}\left(\overset{\circ}{\bigotimes}_{i\in\alpha^{-1}(j)} \mathcal{C}_i, \mathcal{C}'_j\right)$$

with the evident composition law.

– Equip $\mathrm{Cpx}(k[\![\beta]\!])$ with its *projective* model structure, i.e., weak-equivalence and fibrations are maps which are so on underlying complexes. Equipped with $\otimes_{k[\![\beta]\!]}$, it is also a simplicial symmetric-monoidal model category. So $(\mathrm{Perf}\, k[\![\beta]\!], \otimes_{k[\![\beta]\!]})^{\otimes} \to N(\Gamma)^{\otimes}$ admits a description as the homotopy coherent nerve of a fibrant simplicial category $(\mathrm{Cpx}^{\circ,\mathrm{Perf}}(k[\![\beta]\!]))^{\otimes}$ over Γ formed as follows: Its objects are tuples $(\langle n\rangle, C_1, \ldots, C_n)$ with $\langle n\rangle \in \Gamma$ and with each C_i projective-cofibrant perfect dg-$k[\![\beta]\!]$-modules, and simplicial mapping space are given by the same formula as above (with $\bigotimes$ now being taken over $k[\![\beta]\!]$).

– The functor $\mathbf{F}$ preserves fibrant objects, i.e., $\mathbf{F}(V)$ is fibrant for every V. If B acts trivially on V this is clear, since then $\mathbf{F}(V) \simeq V \otimes_k k[\![\beta]\!]$ and $- \otimes_k k[\![\beta]\!]$ is left-adjoint to the forgetful functor which creates fibrations in the projective model structure; the general case reduces to this by writing V as the cone on $(\operatorname{im} B)[-2] \xrightarrow{\beta} \ker B$ and so $\mathbf{F}(V)$ as a cone on projective-cofibrant modules. Furthermore, $\mathbf{F}$ obviously preserves cofibrant objects. We have seen that $\mathbf{F}$ maps complexes with bounded coherent cohomology to perfect complexes. We conclude that there is a well-defined simplicial functor $\mathbf{F}^{\otimes}\colon (\mathrm{Cpx}^{\circ,\mathrm{DCoh}}(\mathcal{O}_{\mathbb{B}}))^{\otimes} \to (\mathrm{Cpx}^{\circ,\mathrm{Perf}}(k[\![\beta]\!]))^{\otimes}$ over $N(\Gamma)$ defined by applying $\mathbf{F}$ to the objects and using the symmetric monoidal structure on the mapping spaces.

– Taking homotopy coherent nerves, we obtain a functor $N(\mathbf{F}^{\otimes}) : (\mathrm{DCoh}(\mathbb{B}), \circ)^{\otimes} \to (\mathrm{Perf}\, k[\![\beta]\!], \otimes_{k[\![\beta]\!]})^{\otimes}$ of coCartesian fibrations over $N(\Gamma)$. To prove that it is a symmetric-monoidal functor, it remains to show that it preserves coCartesian morphisms. The criterion of [L9, Prop. 2.4.1.10] allows us to reduce to showing that if $C_1, \ldots, C_n, D \in \mathrm{Cpx}^{\circ,\mathrm{DCoh}}(\mathcal{O}_{\mathbb{B}})$ and $\bigotimes_i C_i \to D$ is a morphism which induces an equivalence on $\mathrm{Map}(-, E)$ for all $E \in \mathrm{Cpx}^{\circ,\mathrm{DCoh}}(\mathcal{O}_{\mathbb{B}})$, then the same is true for $\bigotimes_i \mathbf{F}(C_i) \to \mathbf{F}(D)$. Since $\bigotimes_i C_i$ is still cofibrant, the first condition is equivalent to the map being a weak-equivalence; since we have seen that $\mathbf{F}$ preserves cofibrant objects, and restricts to fibrant-cofibrant objects, it suffices to observe that it preserves weak-equivalences. □

This yields the promised geometric description of the $k[\![\beta]\!]$-linear structure on $\mathrm{QC}^!$ of a hypersurface:

Construction 3.1.1.5. Suppose $f\colon X \to \mathbb{A}^1$ and set $X_0 = X \times_{\mathbb{A}^1} 0$. Then:

– There is a right action of $\mathbb{B}$ (with its "loop composition") product on X_0. It is easy to give a rigorous Segal-style description via various projections. Heuristically, it is given as follows on (π_0 of) functor of points:

$$X_0(R) \times \mathbb{B}(R) \ni (x \in X(R), [h_f\colon f(x) \to 0] \in R) \times ([h\colon 0 \to 0] \in R)$$
$$\mapsto \left(x \in X(R), [h_f \cdot h\colon f(x) \overset{h_f}{\to} 0 \overset{h}{\to} 0]\right) \in X_0(R)$$

– There is an action of $\mathbb{B}$ (with its "pointwise addition") product on X_0. It is easy to give a rigorous description of it by base-changing the addition map on $\mathbb{A}^1$. Heuristically, it is given as follows on (π_0 of) functor of points:

$$X_0(R) \times \mathbb{B}(R) \ni (x \in X(R), [h_f\colon f(x) \to 0] \in R) \times ([h\colon 0 \to 0] \in R)$$
$$\mapsto (x \in X(R), [h_f + h\colon f(x) \to 0]) \in X_0(R)$$

– As in Construction 3.1.1.2, applying $\mathrm{QC}^!$ to the above group actions equips $\mathrm{QC}^!(X_0)$ with the structure of right $\mathrm{QC}^!(\mathbb{B})$-module (under convolution along loop composition) and compatibly of $\mathrm{QC}^!(\mathbb{B})$-module (under convolution along addition). These are "the same up to homotopy" in the precise sense of the Eckmann-Hilton argument Lemma 3.1.1.6 (c.f., also Remark 3.1.1.7). Note that the structure maps of these actions are *finite* (i.e., affine and finite on π_0): So in fact $\mathrm{DCoh}(X_0)$ is a $\mathrm{DCoh}(\mathbb{B}) = \mathrm{Perf}\, k[\![\beta]\!]$-module, and this recovers the above by passing to Ind-objects.

Lemma 3.1.1.6. *("Eckmann-Hilton") Suppose* $A \in \mathbf{Alg}\,(\mathbf{CAlg}(\mathcal{C}^{\otimes}))$. *Let* $\overline{A} \in \mathbf{CAlg}\,(\mathcal{C}^{\otimes})$ *and* $\widetilde{A} \in \mathbf{Alg}(\mathcal{C})$ *be its images under the forgetful functors. Set* $\mathcal{D}^{\otimes} = \overline{A}\text{-mod}(\mathcal{C}^{\otimes})$, *and note that there is a (lax symmetric monoidal) forgetful functor* $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$. *Then:*

(i) The other commuting product on A *gives rise to a lift of* $\widetilde{A}$ *to an object* $A' \in \mathbf{Alg}\,(\mathcal{D}^{\otimes})$.

(ii) There are equivalences of ∞*-categories*

$$\mathcal{D} \xleftarrow{\sim} A'\text{-mod}\,(\mathcal{D}) \xrightarrow{\sim} \widetilde{A}\text{-mod}\,(\mathcal{C})$$

Remark 3.1.1.7. In the spirit of Remark 3.1.1.3, one can give a similar construction of cosimplicial commutative dg-$\mathcal{O}_X$-algebras encoding the actions of $\mathbb{B}$ on X_0, and the map $X_0/\mathbb{B} \to X$:[3]

$$\left\{ \mathcal{O}_X[x_X, x_1, \ldots, x_\bullet] \begin{bmatrix} \epsilon_1^X, \epsilon_1^1, \ldots, \epsilon_1^\bullet \\ \epsilon_2^X, \epsilon_2^1, \ldots, \epsilon_2^\bullet \end{bmatrix} / \begin{pmatrix} d\epsilon_1^j = d\epsilon_2^j = x \\ d\epsilon_1^X = x - f, d\epsilon_2^X = x \end{pmatrix} \right\} \longleftarrow \mathcal{O}_X$$

$$\left\{ \mathcal{O}_X[x][\gamma_X, \gamma_1, \ldots, \gamma_{\bullet+1}] / \begin{pmatrix} d\gamma_i = x \\ d\gamma_X = x - f \end{pmatrix} \right\} \longleftarrow \mathcal{O}_X[x][\gamma_X]/(d\gamma_X = x - f) \sim \mathcal{O}_X$$

$$\left\{ \mathcal{O}_X[B_X, B_1, \ldots, B_\bullet] / \begin{pmatrix} d(B_i) = 0 \\ d(B_X) = -f \end{pmatrix} \right\} \longleftarrow \mathcal{O}_X$$

In particular, one can avoid explicitly invoking the Eckmann-Hilton argument.

Remark 3.1.1.8. There is an obvious variant of Construction 3.1.1.5 for $\mathrm{QC}(\mathbb{B})$ acting on $\mathrm{QC}(X_0)$. The one notable difference is that this does not pass to compact objects: $\mathrm{Perf}(\mathbb{B})$ is not even monoidal, since the putative tensor unit $\mathcal{O}_{\mathrm{id}} = k$ is not perfect. This is however all that goes wrong: $\mathrm{Perf}(\mathbb{B})$ is an $\otimes$-ideal of $\mathrm{DCoh}(\mathbb{B})$, the inclusion $F_L^{\otimes} \colon \mathrm{QC}(\mathbb{B}) \to \mathrm{QC}^!(\mathbb{B})$ is symmetric-monoidal, and the inclusion $F_L \colon \mathrm{QC}(X_0) \to \mathrm{QC}^!(X_0)$ is linear over $F_L^{\otimes}$ (c.f. Lemma 3.1.1.9). In particular, one may recover the $\mathrm{QC}(\mathbb{B})$-action on $\mathrm{QC}(X_0)$ from the $\mathrm{QC}^!(\mathbb{B})$-action on $\mathrm{QC}^!(X_0)$:

$$V \otimes \mathscr{F} = F_R F_L(V \otimes_{\mathrm{QC}(\mathbb{B})} \mathscr{F}) = F_R \left(F_L^{\otimes}(V) \otimes_{\mathrm{QC}^!(\mathbb{B})} F_L(\mathscr{F}) \right)$$

The relationship between $\mathrm{QC}(\mathbb{B})$ and $\mathrm{QC}^!(\mathbb{B})$ is spelled out by the following Lemma:

Lemma 3.1.1.9. *Under the identification of Prop. 3.1.1.4, the recollement diagram of* $\mathrm{QC}(\mathbb{B})$, $\mathrm{QC}^!(\mathbb{B})$, $\mathrm{DSing}^\infty(\mathbb{B})$ *associated to the Drinfeld-Verdier sequence*[4]

$$\mathrm{Perf}(\mathbb{B}) = \mathrm{Perf}\,\mathcal{O}_{\mathbb{B}} \xrightarrow{F} \mathrm{DCoh}(\mathbb{B}) = \mathrm{Perf}\,k[\![\beta]\!] \xrightarrow{G} \mathrm{DSing}(\mathbb{B}) = \mathrm{Perf}\,k(\!(\beta)\!)$$

may be identified with

$$k(\!(\beta)\!)\text{-mod} \;\; \begin{matrix} \xleftarrow{\quad G_L \quad} \\ \xrightarrow{\quad G_M \quad} \\ \xleftarrow{\quad G_R \quad} \end{matrix} \;\; k[\![\beta]\!]\text{-mod} \;\; \begin{matrix} \xleftarrow{\quad F_L \quad} \\ \xrightarrow{\quad F_M \quad} \\ \xleftarrow{\quad F_R \quad} \end{matrix} \;\; \mathcal{O}_{\mathbb{B}}\text{-mod}$$

where

- $F_L = (-)_{S^1}[1] = k[1] \otimes_{\mathcal{O}_B} -$;
- $F_M = \mathrm{RHom}_{k[\![\beta]\!]}(k, -) = k \otimes_{k[\![\beta]\!]} -$;
- $F_R = (-)^{S^1} = \mathrm{RHom}_{\mathcal{O}_B}(k, -)$;
- $G_L = k(\!(\beta)\!) \otimes_{k[\![\beta]\!]} -$;

[3] There is a choice of sign appearing here that will probably change at random later.↑

[4] In dg-category language the functors in the recollement diagram are restriction, induction, and co-induction of right dg-modules over the terms in the Drinfeld-Verdier sequence; i.e., F_M is the restriction along F, F_L is induction along F, and F_R is coinduction along F.↑

– $G_M = \mathrm{RHom}_{k(\!(\beta)\!)}(k(\!(\beta)\!), -) = k(\!(\beta)\!) \otimes_{k(\!(\beta)\!)} -$;

– $G_R = \mathrm{RHom}_{k[\![\beta]\!]}(k(\!(\beta)\!), -)$.

In particular, these satisfy all the usual relations (e.g., the unit $\mathrm{id} \to F_M \circ F_L$ *and the counit* $F_M \circ F_R \to \mathrm{id}$ *are equivalences, etc.) so that* F_L *induces an equivalence*

$$F_L \colon \mathrm{QC}(\mathbb{B}) \xrightarrow{\sim} \left\{ \begin{matrix} \textit{locally } \beta\textit{-torsion} \\ k[\![\beta]\!]\textit{-modules} \end{matrix} \right\} \overset{def}{=} \mathrm{Ind} \begin{pmatrix} \beta\textit{-torsion perfect} \\ k[\![\beta]\!]\textit{-modules} \end{pmatrix}$$

Proof. We first focus only on the F side: The various functors have the right adjunctions simply by Morita theory, so the identification follows from noting that F_L does the right thing on compact objects; the coincidence of two descriptions for F_M is a straightforward computation. The G side will follow similarly once we show identify $G \colon \mathrm{DCoh}(\mathbb{B}) \to \mathrm{DSing}(\mathbb{B})$ with $k(\!(\beta)\!) \otimes_{k[\![\beta]\!]} - \colon \mathrm{Perf}\, k[\![\beta]\!] \to \mathrm{Perf}\, k(\!(\beta)\!)$. The description of $F_L(\mathrm{QC}(\mathbb{B}))$ as locally β-torsion β-modules then follows from the description of G_L.

The cofiber sequence

$$k[\![\beta]\!][-2] \xrightarrow{t} k[\![\beta]\!] \to k$$

identifies $\mathrm{DSing}^\infty(\mathbb{B})$ (=the fiber of F_M) with the full-subcategory of $k[\![\beta]\!]$-mod consisting of objects $\mathscr{F}$ on which $t \colon \mathscr{F}[-2] \to \mathscr{F}$ is an equivalence. This can be identified with the ∞-category of $k(\!(\beta)\!)$-module objects in $k[\![\beta]\!]$-mod: For any such $\mathscr{F}$, the natural map

$$k(\!(\beta)\!) \otimes_{k[\![\beta]\!]} \mathscr{F} = \varinjlim_n \frac{1}{\beta^n} \mathscr{F} \longrightarrow \mathscr{F}$$

is an equivalence (since the lim is taken over a diagram of equivalences). Finally, the adjunction (G_L, G_R) is monadic and identifies $k(\!(\beta)\!)$-mod with $k(\!(\beta)\!)$-module objects in $k[\![\beta]\!]$. Passing to compact objects gives the desired identification. □

Construction 3.1.1.5 tells us that any $\mathscr{K} \in \mathrm{QC}^!(\mathbb{B})$ gives rise to an endo-functor of $\mathrm{QC}^!(X_0)$. We'll spell this out for several distinguished objects of $\mathrm{QC}^!(\mathbb{B})$.

Example 3.1.1.10. Consider $\mathcal{O}_{\mathbb{B}} \in \mathrm{Perf}(\mathbb{B})$, i.e., $k[B]/B^2$ as a perfect $k[B]/B^2$-module. Since it is perfect, $F_L(\mathcal{O}_{\mathbb{B}}) = F_R(\mathcal{O}_{\mathbb{B}})$ and both are identified under the equivalence of Prop. 3.1.1.4 with

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}}(k, \mathcal{O}_{\mathbb{B}}) \simeq k[1] \in k[\![\beta]\!]\text{-mod}$$

Base-change in the Cartesian diagram (and the "loop composition" description of the action)

$$\begin{array}{ccc} X_0 \times \mathbb{B} & \xrightarrow{\alpha} & X_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle i} \\ X_0 & \xrightarrow[i]{} & X \end{array}$$

implies $\mathcal{O}_{\mathbb{B}} \otimes_{k[\![\beta]\!]} -$ may be identified with $i^* i_* \colon \mathrm{QC}^!(X_0) \to \mathrm{QC}^!(X_0)$. (This makes sense on each of QC, $\mathrm{QC}^!$, and DCoh since i is finite and of finite Tor-dimension. The functor on QC restricts to one on DCoh, and the functor on $\mathrm{QC}^!$ is then the Ind-extension of this.)

Example 3.1.1.11. The object $\mathcal{O}_{\mathbb{B}} \in \mathrm{QC}(\mathbb{B})$ admits an S^1-action ("multiplication by B", i.e., the S^1-action on $C_*(S^1;k)$), equipped with equivalences

$$(\mathcal{O}_{\mathbb{B}})_{S^1} = \mathcal{O}_{\mathrm{id}} \quad (\text{i.e., } C_*(S^1;k)_{S^1} = C_*(S^1/S^1;k))$$

$$(\mathcal{O}_{\mathbb{B}})^{S^1} = \mathcal{O}_{\mathrm{id}}[1] \quad (\text{i.e., } C_*(S^1;k)^{S^1} = C_*(S^1/S^1;k)[1])$$

The normalized chain complex of the simplicial-bar construction computing the homotopy quotient is the Koszul-Tate resolution used in the proof of Prop. 3.1.1.4:

$$\mathcal{O}_{\mathrm{id}} \xleftarrow{\sim} \mathcal{O}_B \left[\underbrace{u^k/k!}_{\deg u = +2} \right] /du = B \qquad \in \mathrm{QC}(\mathbb{B})$$

By Construction 3.1.1.5 this gives rise to an explicit equivalence of endo-functors on $\mathrm{QC}(X_0)$

$$(i^* i_*)_{S^1} \simeq i^* i_*[u^k/k!] \sim \mathrm{id}_{\mathrm{QC}(X_0)}$$

i.e., a natural equivalence

$$(i^*i_*)_{S^1}\mathscr{F} = \left|(i^*i_*)^{\bullet+1}\mathscr{F}\right| \simeq (i^*i_*\mathscr{F})\left[u^k/k!\right] = \mathrm{Tot}^{\oplus}\left\{\cdots i^*i_*\mathscr{F}[2] \xrightarrow{B} i^*i_*\mathscr{F}[1] \xrightarrow{B} i^*i_*\mathscr{F}\right\} \xrightarrow{\sim} \mathscr{F}$$

for $\mathscr{F} \in \mathrm{QC}(X_0)$.

Example 3.1.1.12. Consider $\mathcal{O}_{\mathrm{id}} = \Delta_*\mathcal{O}_{\mathrm{pt}} \in \mathrm{DCoh}(\mathbb{B}) \subset \mathrm{QC}(\mathbb{B})$. It gives rise to a natural cofiber sequence in $\mathrm{QC}^!(\mathbb{B})$

$$F_L(\mathcal{O}_{\mathrm{id}}) \longrightarrow F_R(\mathcal{O}_{\mathrm{id}}) \longrightarrow \mathrm{cone}\left\{F_L(\mathcal{O}_{\mathrm{id}}) \to F_R(\mathcal{O}_{\mathrm{id}})\right\}$$

Under the identification of Prop. 3.1.1.4, this sequence may be identified (c.f., Example 3.1.1.11) with the *Tate sequence* $k_{S^1}[1] \to k^{S^1} \to k^{\mathrm{Tate}}$:

$$\underbrace{k(\!(\beta)\!)/k[\![\beta]\!][-1]}_{\mathcal{O}_B[u^k/k!]} \longrightarrow \underbrace{k[\![\beta]\!]}_{\mathcal{O}_{\mathrm{id}}} \longrightarrow k(\!(\beta)\!)$$

The three act as follows: $\mathcal{O}_{\mathrm{id}} = F_R(\mathcal{O}_{\mathrm{id}})$ acts by the identity on $\mathrm{QC}^!(M_0)$ (it had better, it is the tensor unit); $k(\!(\beta)\!)$ acts by inverting β; $F_L(\mathcal{O}_{\mathrm{id}})$ acts by the colimit/simplicial diagram of Example 3.1.1.11, which coincides with the identify functor on $\mathrm{QC}(M_0)$ but not in general.

Example 3.1.1.13. The Postnikov filtration of $k[B]/B^2$ yields a fiber sequence

$$\underbrace{F_R(k[1])}_{k[\![\beta]\!][1]} \longrightarrow \underbrace{F_R(k[B]/B^2)}_{k[1]} \longrightarrow \underbrace{F_R(k)}_{k[\![\beta]\!]}$$

which is just (a rotation of) the identification $k[1] = \mathrm{cone}\{t\colon k[\![\beta]\!][-1] \to k[\![\beta]\!][1]\}$. By Construction 3.1.1.5 this gives rise to a fiber sequence of functors $\mathrm{id}[1] \to i^*i_* \to \mathrm{id}$. More explicitly, for any $\mathscr{F} \in \mathrm{QC}^!(M_0)$ there is a triangle

$$\mathscr{F}[1] \longrightarrow i^*i_*\mathscr{F} \longrightarrow \mathscr{F} \to$$

where the second map is the counit of the adjunction (recall i is of finite Tor-dimension).

3.1.2 Circle actions

Proposition 3.1.2.1. *Suppose $\mathscr{F}, \mathscr{G} \in \mathrm{PreMF}(M, f)$. Let $i\colon M_0 \to M$ be the inclusion. Then, there is a natural circle action on $i^* i_* \mathscr{F}$ and a natural equivalence $(i^* i_* \mathscr{F})_{S^1} = \mathscr{F}$. This gives rise, by adjunction, to natural S^1 actions on $\mathrm{Map}(i_*\mathscr{F}, i_*\mathscr{G})$ and $\mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})$ such that*

- *There is a natural equivalence $\mathrm{Map}(\mathscr{F}, \mathscr{G}) = \mathrm{Map}(i_*\mathscr{F}, i_*\mathscr{G})^{S^1}$;*
- *There is a natural $k[\![\beta]\!]$-linear equivalence*
$$\mathrm{RHom}_{M_0}^{\otimes k[\![\beta]\!]}(\mathscr{F}, \mathscr{G}) = \mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})^{S^1}$$
- *Let $\overline{\mathscr{F}}, \overline{\mathscr{G}} \in \mathrm{MF}(M, f)$ denote the images of $\mathscr{F}, \mathscr{G}$. Then, there is a natural $k(\!(\beta)\!)$-linear equivalence*
$$\mathrm{RHom}_{\mathrm{MF}(M,f)}^{\otimes k(\!(\beta)\!)}(\overline{\mathscr{F}}, \overline{\mathscr{G}}) = \mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})^{\mathrm{Tate}}$$

Proof. This follows from Example 3.1.1.11, which gives an S^1 action on $i^*i_* \in \mathrm{Fun}_k^L(\mathrm{QC}^!(M_0), \mathrm{QC}^!(M_0))$ with $(i^*i_*)_{S^1} = \mathrm{id}$. Since i^*i_* preserves $\mathrm{DCoh}(M_0)$, the indicated equivalence restricts.

Let us spell things out more explicitly: The simplicial bar construction computing $(\mathcal{O}_B)_{S^1}$ identifies under Dold-Kan with the Koszul-Tate resolution of Example 3.1.1.11. Thus we have a functorial equivalence in $\mathrm{QC}(M_0)$

$$\left[(i^*i_*\mathscr{F})[u^k/k!]/du = B\right] = \mathrm{Tot}^{\oplus}\left\{\cdots \xrightarrow{B} \Sigma^2 i^*i_*\mathscr{F} \xrightarrow{B} \Sigma i^*i_*\mathscr{F} \xrightarrow{B} i^*i_*\mathscr{F}\right\} \longrightarrow \mathscr{F}$$

which, since $\mathrm{PreMF}(M, f) = \mathrm{DCoh}(M_0)$ is a full subcategory of $\mathrm{QC}(M_0)$, gives rise to an equivalence

$$\begin{aligned}\mathrm{Map}_{\mathrm{PreMF}(M,f)}(\mathscr{F}, \mathscr{G}) &= \mathrm{Tot}\left\{\mathrm{Map}_{\mathrm{DCoh}(M_0)}(i^*i_*\mathscr{F}, \mathscr{G})\right\} \\ &= \mathrm{Tot}\left\{\mathrm{Map}_{\mathrm{DCoh}(M)}(i_*\mathscr{F}, i_*\mathscr{G})\right\} \\ &= \mathrm{Map}_{\mathrm{DCoh}(M)}(i_*\mathscr{F}, i_*\mathscr{G})^{S^1}\end{aligned}$$

where the S^1-action is given by B. (See below for a yet more explicit form.) □

Remark 3.1.2.2. For the simplicially-inclined reader, we mention the following alternate description: For $\mathscr{F} \in \mathrm{QC}(M_0)$, there is an augmented, i_*-split, simplicial object

$$\{(i^*i_*)^\bullet \mathscr{F}\} = \left\{\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (i^*i_*)^2\mathscr{F} \rightrightarrows i^*i_*\mathscr{F}\right\} \longrightarrow \mathscr{F}$$

which realizes $\mathscr{F}$ as the geometric realization of the simplicial diagram (c.f., Lemma 3.1.3.1). Identifying $\mathcal{O}_{\mathbb{B}} = C_*(S^1, k)$, the diagram encodes an S^1-action on $i^*i_*\mathscr{F} \in \mathrm{QC}(M_0)$, with quotient $(i^*i_*\mathscr{F})_{S^1} = \mathscr{F}$. For $\mathscr{F} \in \mathrm{DCoh}(M_0)$ there is also a Grothendieck-dual description (c.f., Example 3.1.1.12)

$$\mathscr{F} \longrightarrow \left\{(i^! i_*)_\bullet \mathscr{F}\right\} \simeq \{(i^*i_*)_\bullet \mathscr{F}[-1]\}$$

Lemma 3.1.2.3. *Suppose V is a complex with S^1-action. Then, the natural map*

$$V^{S^1} \otimes_{k[\![\beta]\!]} k \longrightarrow V$$

is an equivalence.

Proof. Identify the ∞-category of complexes with S^1-action with $k[B]/B^2$-mod. Identify $k[\![\beta]\!] = \mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}(k,k)$. Then, the map in question is identified with the natural evaluation map

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{B}}\text{-mod}}(k,V) \otimes_{k[\![\beta]\!]} k \longrightarrow V$$

i.e., the counit $F_M \circ F_R \to \mathrm{id}$. This is an equivalence by Lemma 3.1.1.9. $\square$

Corollary 3.1.2.4. *Suppose $Z \subset M_0$ is a closed subset. Then,* $i_*\colon \mathrm{DCoh}_Z(M_0) \to \mathrm{DCoh}_Z(M)$ *induces an equivalence of ∞-categories*

$$i_*\colon \mathrm{PreMF}_Z(M,f) \otimes_{k[\![\beta]\!]} k \xrightarrow{\sim} \mathrm{DCoh}_Z M$$

Proof. First, we will construct the desired lift of i_* via a geometric description of the simplicial bar construction implementing the tensor product: The augmented simplicial diagram

$$\left\{M_0 \times \mathbb{B}^{\bullet-1} \times \mathrm{pt}\right\} = \left\{\cdots\ M_0 \times \mathbb{B} \times \mathrm{pt} \underset{p_1}{\overset{\alpha}{\rightrightarrows}} M_0 \times \mathrm{pt}\right\} \xrightarrow{i} M$$

gives rise to an augmented simplicial diagram of ∞-categories, which via Prop. A.2.3.2 may be identified with

$$\left\{\mathrm{DCoh}(M_0) \otimes \mathrm{DCoh}(\mathbb{B})^{\otimes \bullet-1} \otimes \mathrm{DCoh}(\mathrm{pt})\right\} \xrightarrow{i_*} \mathrm{DCoh}(M)$$

where the simplicial diagram is the simplicial bar construction for $\mathrm{PreMF}(M,f) \otimes_{\mathrm{DCoh}(\mathbb{B})} \mathrm{DCoh}(\mathrm{pt}) = \mathrm{PreMF}(M,f) \otimes_{k[\![\beta]\!]} k$. Imposing support conditions everywhere, we obtain the functor of the statement.

Next we verify that this functor is fully faithful: It suffices to check that for any $\mathscr{F}, \mathscr{G} \in \mathrm{PreMF}(M,f)$ the natural map

$$\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}(M,f)}(\mathscr{F},\mathscr{G}) \otimes_{k[\![\beta]\!]} k \longrightarrow \mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})$$

is an equivalence. This follows immediately from the triangle of Example 3.1.1.13 and adjunction.

Alternatively, by Prop. 3.1.2.1 we may identify this with the map

$$\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}(M,f)}(\mathscr{F},\mathscr{G}) \otimes_{k[\![\beta]\!]} k = \mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})^{S^1} \otimes_{k[\![\beta]\!]} k \longrightarrow \mathrm{RHom}_M(i_*\mathscr{F}, i_*\mathscr{G})$$

which is an equivalence by Lemma 3.1.2.3.

We now prove that the functor is essentially surjective: Since it is fully faithful, and $\mathrm{DCoh}_Z M$ is a sheaf on M, the question is local and we may suppose M is a quasi-compact coherent scheme. Since both sides are stable and idempotent complete, it suffices to show that it has dense image. We conclude by noting that $i_*\colon \mathrm{DCoh}_Z(M_0) \to \mathrm{DCoh}_Z M$ has dense image by the usual t-structure and filtration argument, since $Z \subset M_0$ (c.f., Lemma 2.2.0.2). $\square$

3.1.3 Computational tools

The following Lemma is rather categorical, and may be safely skipped on first reading: we will try to extract and emphasize its more concrete consequences.

Lemma 3.1.3.1. *Suppose X', X are coherent derived stacks, that $i\colon X' \to X$ is finite and of finite Tor-dimension. Then,*

(i) The adjoint pair $i^*\colon \mathrm{QC}(X) \rightleftarrows \mathrm{QC}(X')\colon i_*$ is monadic, and induces an equivalence

$$\mathrm{QC}(X') \simeq (i_*\mathcal{O}_{X'})\text{-mod}\,(\mathrm{QC}(X))$$

(ii) The adjunction of (i) restricts to an adjoint pair $i^*\colon \mathrm{DCoh}(X) \rightleftarrows \mathrm{DCoh}(X')\colon i_*$. It is also monadic, and induces an equivalence

$$\mathrm{DCoh}(X') \simeq i_*\mathcal{O}_{X'}\text{-mod}\,(\mathrm{DCoh}(X))$$

Proof.

(i) Since i is affine, i_* admits a Cech description and hence preserves all colimits. Furthermore, i_* is conservative: The question is local, and the result is true for X affine because i^* will pull back a generator to a generator. Since $\mathrm{QC}(X')$ has all colimits, Lurie's Barr-Beck Theorem applies to give $\mathrm{QC}(X') \simeq i_*i^*\text{-mod}\,\mathrm{QC}(X)$ and this monad visibly identifies with that given by the algebra $i_*i^*\mathcal{O}_X = i_*\mathcal{O}_{X'}$.

(ii) Since i is assumed finite, i_* preserves DCoh. Since i is assumed of finite Tor-dimension, i^* preserves DCoh. So, the adjunction restricts to one on the full subcategories $i^*\colon \mathrm{DCoh}(X) \rightleftarrows \mathrm{DCoh}(X')\colon i_*$. It thus suffices to show that for $\mathscr{G} \in \mathrm{QC}(X')$, we have $\mathscr{G} \in \mathrm{DCoh}(X')$ iff $i_*\mathscr{G} \in \mathrm{DCoh}(X)$. Since i is affine, i_* is t-exact; that is,

$$i_*^{\heartsuit}\left(\pi_m\mathscr{G}\right) = \pi_m\left(i_*\mathscr{G}\right) \in \mathrm{QC}(X)^{\heartsuit} = \pi_0(\mathcal{O}_X)\text{-mod}$$

It remains to show that

$$i_*^{\heartsuit}\colon \mathrm{QC}(X')^{\heartsuit} \to \mathrm{QC}(X)^{\heartsuit}$$

is conservative, and that $i_*^{\heartsuit}(M)$ is finitely-presented over $\pi_0(\mathcal{O}_X)$ iff M is finitely-presented over $\pi_0(\mathcal{O}_{X'})$.

The question is local on X, so we will assume $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$. Note that $i_0\colon \operatorname{Spec}\pi_0 B \to \operatorname{Spec}\pi_0 A$ is a finite map of discrete coherent rings, and that we must show that the pushforward $(i_0)_*^{\heartsuit}$ of discrete modules is conservative, and that it preserves and detects the property of a module being finitely-presented. That it is conservative is obvious. Since $\pi_0 B$ is finite over $\pi_0 A$, it preserves the property of being finitely-presented. To see that it detects coherence, suppose M is a $\pi_0 B$-module such that the corresponding $\pi_0 A$-module, denoted M_A, is coherent. Considering the surjection $M_A \otimes_{\pi_0 A} \pi_0 B \to M$, we see that M is finitely-generated, so that if $\pi_0 B$ is Noetherian we are done. To handle the general coherent case, we reduce to the case of $\pi_0 A$ (and so $\pi_0 B$) Noetherian: It suffices to note that we may find a Noetherian subring of $\pi_0 A$ over which the coherent $\pi_0 A$-algebra $\pi_0 B$, the coherent $\pi_0 A$-module M_A, and the $\pi_0 B$-action on M_A, are all defined. □

Corollary 3.1.3.2. *Suppose (M, f) is an LG pair. Set*

$$\mathcal{A} = \left(\underbrace{\mathcal{O}_M[B_M]}_{\deg B_M = +1} \Big/ \begin{array}{c} B_M^2 = 0 \\ dB_M = f \end{array} \right)$$

so that $\mathcal{A}$ is an algebra. Then, (i_*, i^*) induces equivalences

$$\mathrm{QC}(M_0) = \mathcal{A}\text{-mod}\,(\mathrm{QC}(M)) \qquad \mathrm{DCoh}(M_0) = \mathcal{A}\text{-mod}\,(\mathrm{Perf}(M))$$

3.1.3.3. Under the identification of Cor. 3.1.3.2, we can make explicit the $k[\![\beta]\!]$-linear structure of Prop. 3.1.1.4(iii) as follows: Suppose $\mathscr{F} \in \mathcal{A}\text{-mod}(\mathrm{QC}(M))$. The resolution of Example 3.1.1.11 is

$$\mathscr{F} \sim \underbrace{\mathscr{F}[B_{\mathrm{new}}][u^\ell/\ell!]}_{\deg B_{\mathrm{new}} = +1, \deg u = +2} \Big/ \left\{ \begin{array}{c} B_{\mathrm{new}}^2 = 0 \\ dB_{\mathrm{new}} = f \\ du = B_{\mathrm{new}} - B_{\mathscr{F}} \end{array} \right\}$$

where B_M acts on the RHS by B_{new} (in particular, $\mathscr{F}$ is not a submodule). The $k[\![\beta]\!]$ action on the right hand-side is given by $\beta = d/du$. For $\mathscr{F} \in \mathcal{A}\text{-mod}(\mathrm{Perf}\, M)$ there is also the Grothendieck dual resolution (c.f., Remark 3.1.2.2)

$$\mathscr{F} \sim (\mathscr{F}[B_{\mathrm{new}}][\![\beta]\!], d_{\mathscr{F}} + (B_{\mathrm{new}} - B_{\mathscr{F}})\beta)$$

on which B_M acts as above and the β action is evident.

For concreteness, let us also make explicit the Dold-Kan computation mentioned several times above: The augmented simplicial object of Example 3.1.1.11 can be identified with

$$\mathscr{F} \xleftarrow{\partial^0_{-1}} \left\{ \mathscr{F}[B_0] \; \begin{array}{c} \xleftarrow{\partial^1_0} \\ \xleftarrow{\partial^0_0} \end{array} \; \mathscr{F}[B_0, B_1] \; \begin{array}{c} \xleftarrow{\partial^2_1} \\ \xleftarrow{\partial^1_1} \\ \xleftarrow{\partial^0_1} \end{array} \; \mathscr{F}[B_0, B_1, B_2] \; \cdots \right\}$$

with the face maps determined, via the Liebniz rule, by

$$\partial^i_n \colon \mathscr{F}[B_0, \ldots, B_{n+1}] \to \mathscr{F}[B_0, \ldots, B_n] \qquad B_k \mapsto \begin{cases} B_k & \text{if } k < i \\ B_{k-1} & \text{otherwise} \end{cases}$$

where by convention $B_{-1} = B_{\mathscr{F}}$ denotes acting by B_M on $\mathscr{F}$. This determines a simplicial object in the abelian category of dg-$\mathcal{O}_M[B_M]$-modules, to which we apply the Dold-Kan correspondence by forming the normalized chain complex:

$$N_\ell = \bigcap_{i \geq 1} \ker \partial^i_{\ell-1} = \mathscr{F} \cdot \left(\sum_{i=0}^{\ell} (-1)^{\ell-i} B_0 \cdots \widehat{B_i} \cdots B_\ell \right) \oplus \mathscr{F} \cdot (B_0 B_1 \cdots B_\ell)$$

and with respect to these direct sum decompositions, the differential takes the form

$$\partial^0_\ell = \begin{pmatrix} B_{\mathscr{F}} & 0 \\ (-1)^\ell & B_{\mathscr{F}} \end{pmatrix}$$

One checks that (at least with an appropriate sign convention), the totalization of this coincides with the above "Koszul-Tate" complex under the identifications

$$B_{\text{new}} \mapsto B_0 \qquad \text{and} \qquad \frac{u^\ell}{\ell!} \mapsto \sum_{i=0}^{\ell} (-1)^{\ell-i} B_0 \cdots \widehat{B_i} \cdots B_\ell$$

3.1.3.4. Under the identification of Cor. 3.1.3.2, we can make explicit the S^1-action of Prop. 3.1.2.1 as follows: Adjunction provides an equivalence

$$\operatorname{RHom}_{\mathcal{O}_M}(\mathscr{F}, \mathscr{G}) = \operatorname{RHom}_{\mathcal{O}_M[B_M]}(\mathscr{F}[B], \mathscr{G}) \qquad \phi\,(f) \mapsto \widetilde{\phi}\,(f + Bf') = \phi\,(f - B_M \cdot f') + B_M \phi\,(f')$$

On the right hand side, the B-operator of the circle action is the (graded) dual of multiplication by B. A straightforward computation then shows that the induced operation on the left-hand side is (at least up to signs)

$$B\phi = B_M \circ \phi + \phi \circ B_M$$

3.1.4 Comparison of definitions

Proposition 3.1.4.1. *Suppose (M, f) is an LG pair. Then, the natural functor*

$$\operatorname{PreMF}(M, f) \longrightarrow \operatorname{MF}(M, f) \overset{def}{=} \operatorname{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$$

factors through the quotient functor $\operatorname{DCoh}(M_0) \to \operatorname{DSing}(M_0) = \operatorname{DCoh}(M_0)/\operatorname{Perf}(M_0)$. *The induced functor*

$$\operatorname{DSing}(M_0) \longrightarrow \operatorname{MF}(M, f)$$

is an idempotent completion.

Proof. This can be found in the literature, but as this is important for our approach we sketch an argument.
Claim: Suppose $\mathscr{F} \in \operatorname{DCoh}(M_0)$. Then, TFAE

(i) $\mathscr{F}$ is perfect;

(ii) $\operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{DCoh}(M_0)}(\mathscr{F}, \mathscr{F})$ is β-torsion (i.e., there is an $N > 0$ such that β^N is null-homotopic);

(iii) $\operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{DCoh}(M_0)}(\mathscr{F}, \mathscr{F})$ is locally β-torsion (i.e., it is a filtered colimit of perfect t-torsion $k[\![\beta]\!]$-modules);

(iv) $1 \in \pi_* \operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{DCoh}(M_0)}(\mathscr{F}, \mathscr{F})$ is β-torsion;

(v) $\mathscr{F} \mapsto 0 \in \operatorname{MF}(M, f)$.

Assuming the claim, we complete the proof. The existence of a factorization through a functor $\operatorname{DSing}(M_0) \to \operatorname{MF}(M, f)$ follows from (iv) by the universal property of a Drinfeld-Verdier quotient (as cofiber in small k-linear ∞-categories). Since the image of $\operatorname{DCoh}(M_0)$ is dense (i.e., its thick closure is the whole) in both, it suffices to show that this functor is fully-faithful. More precisely, it suffices to show that for $\mathscr{F}, \mathscr{G} \in \operatorname{DCoh}(M_0)$ the natural map

$$\pi_0 \operatorname{RHom}^{\otimes k(\!(\beta)\!)}_{\operatorname{DSing}(M_0)}(\overline{\mathscr{F}}, \overline{\mathscr{G}}) \to \pi_0 \left[\operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{DCoh}(M_0)}(\mathscr{F}, \mathscr{G}) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)\right]$$

is an equivalence.

At this point, we may conclude in several ways:

– (Lazy) We may identify

$$\begin{aligned}\pi_0 \left[\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F},\mathscr{G}) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)\right] &= \pi_0 \varinjlim_n \left[\frac{1}{\beta^n} \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0}(\mathscr{F},\mathscr{G})\right] \\ &= \varinjlim_n \pi_0 \left[\frac{1}{\beta^n} \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0}(\mathscr{F},\mathscr{G})\right] \\ &= \varinjlim_n \pi_{-2n} \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0}(\mathscr{F},\mathscr{G})\end{aligned}$$

However, the following formula for maps in $\pi_0 \, \mathrm{DSing}(M_0)$ appears in the literature

$$\pi_0 \, \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{DSing}(M_0)}(\overline{\mathscr{F}},\overline{\mathscr{G}}) = \varinjlim_n \pi_{-2n} \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F},\mathscr{G})$$

and one may check that it is induced by the map we wrote down above.

– (Less lazy) Consider applying $\mathrm{DCoh}(M_0) \otimes_{\mathrm{DCoh}(\mathbb{B})} -$ to the (idempotent completed) Drinfeld-Verdier sequence

$$\mathrm{Perf}(\mathbb{B}) \to \mathrm{DCoh}(\mathbb{B}) \to \mathrm{DSing}(\mathbb{B})$$

which by Lemma 3.1.1.9 may be identified with

$$\left\{ \begin{matrix} \beta\text{-torsion} \\ \text{perfect } k[\![\beta]\!]\text{-mod} \end{matrix} \right\} \longrightarrow \mathrm{Perf}\, k[\![\beta]\!] \xrightarrow{-\otimes_{k[\![\beta]\!]}} \mathrm{Perf}\, k(\!(\beta)\!)$$

The result will again be an (idempotent completed) Drinfeld-Verdier sequence (Lemma 3.1.4.2). The Claim implies that

$$\mathrm{DCoh}(M_0) \otimes_{\mathrm{DCoh}(\mathbb{B})} \mathrm{Perf}(\mathbb{B}) = \mathrm{Perf}(M_0)$$

Indeed, the LHS identifies with the full subcategory of $\mathrm{DCoh}(M_0)$ consisting of objects with locally β-torsion endomorphisms. Since the first two terms of a Drinfeld-Verdier sequence determine the third up to idempotent completion, this completes the proof.

Proof of Claim: Recall the resolution in $\mathrm{QC}(M_0)$ (Example 3.1.1.11)

$$\mathrm{hocolim}\,\{\cdots \to i^* i_* \mathscr{F}[2] \to i^* i_* \mathscr{F}[1] \to i^* i_* \mathscr{F}\} \xrightarrow{\sim} \mathscr{F}$$

Computing $\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F},\mathscr{F})$ using this resolution, we see that for $N > 0$ a null-homotopy of β^N on $\mathrm{RHom}^{\otimes k[\![\beta]\!]}(\mathscr{F},\mathscr{F})$ realizes $\mathscr{F}$ as a homotopy retract of

$$\mathrm{hocolim}\,\{i^* i_* \mathscr{F}[N] \to \cdots \to i^* i_* \mathscr{F}\}$$

and conversely if $\mathscr{F}$ is a homotopy retract of this then β^N is null-homotopic on $\mathrm{RHom}^{\otimes k[\![\beta]\!]}(\mathscr{F},\mathscr{F})$.

If $\mathscr{F} \in \mathrm{Perf}(M_0)$, then it is compact in $\mathrm{QC}(M_0)$ and the identity factors through a finite piece as above. Thus (i) implies (ii).

Conversely: $i_*\mathscr{F}$ is coherent since i is finite, and thus perfect since M is regular. Thus $i^*i_*\mathscr{F}$ is perfect, and so is anything built from it by shifts, finite colimits, and retracts. This proves that (ii) implies (i).

The implication (ii) implies (iii) is immediate.

Note that $\mathcal{A} = \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F}, \mathscr{F})$ is a $k[\![\beta]\!]$-algebra, and consider the unit $1\colon k[\![\beta]\!] \to \mathcal{A}$. If $\mathcal{A}$ is locally t-torsion, then we may write $\mathcal{A} = \varinjlim P_\alpha$ with P_α perfect and t-torsion; since $k[\![\beta]\!]$ is perfect, 1 factors through $1\colon k[\![\beta]\!] \to P_\alpha$ for some α. Consequently, there exists $N > 0$ so that $\beta^N \cdot 1\colon k[\![\beta]\!][2N] \to P_\alpha$, and so also $\beta^N \cdot 1\colon k[\![\beta]\!][2N] \to \mathcal{A}$, is null-homotopic. This implies that $\beta^N \cdot 1 = 0 \in \pi_{-2n}\mathcal{A}$, proving that (iii) implies (iv).

Conversely, if $\beta^N \cdot 1 = 0 \in \pi_{-2N}\mathcal{A}$ then $\beta^N \cdot 1\colon k[\![\beta]\!][2N] \to \mathcal{A}$ is null-homotopic. Since $\mathcal{A}$ is an algebra, we conclude that $\beta^N\colon \mathcal{A}[2N] \to \mathcal{A}$ is null-homotopic. This proves that (iv) implies (ii).

Finally, note that $\mathscr{F} \mapsto 0 \in \mathrm{MF}(M, f)$ if and only if $1 = 0 \in \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}(M,f)}(\mathscr{F}, \mathscr{F})$. Since

$$\begin{aligned}\pi_0 \,\mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}(M,f)}(\mathscr{F}, \mathscr{F}) &= \pi_0 \varinjlim_N \frac{1}{\beta^N} \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F}, \mathscr{F}) \\ &= \varinjlim_N \pi_{-2N} \,\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{DCoh}(M_0)}(\mathscr{F}, \mathscr{F})\end{aligned}$$

where the filtered limit is formed under multiplication by β. This proves that (iv) $\Leftrightarrow$ (v). ☐

Lemma 3.1.4.2.

(i) Suppose $\mathcal{A}^\otimes \in \mathbf{CAlg}(\mathbf{dgcat}^{\mathrm{idm}}_k)$ is a rigid symmetric-monoidal dg-category, $\mathcal{D} \in \mathcal{A}\text{-mod}(\mathbf{dgcat}^{\mathrm{idm}}_k)$ an $\mathcal{A}$-module category, and

$$\mathcal{C}' \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}''$$

a diagram in $\mathcal{A}\text{-mod}(\mathbf{dgcat}^{\mathrm{idm}}_k)$ which is a Drinfeld-Verdier sequence. Then,

$$\mathcal{C}' \otimes_{\mathcal{A}} \mathcal{D} \longrightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{D} \longrightarrow \mathcal{C}'' \otimes_{\mathcal{A}} \mathcal{D}$$

is again a Drinfeld-Verdier sequence.

(ii) Suppose $R \in \mathbf{CAlg}(k\text{-mod})$ is a commutative dg-k-algebra, $\mathcal{D} \in \mathbf{dgcat}^{\mathrm{idm}}_R$ an R-linear dg-category, and

$$\mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}''$$

an R-linear Drinfeld-Verdier sequence. Then,

$$\mathcal{C}' \otimes_R \mathcal{D} \longrightarrow \mathcal{C} \otimes_R \mathcal{D} \longrightarrow \mathcal{C}'' \otimes_R \mathcal{D}$$

is again a Drinfeld-Verdier sequence.

Proof. For (ii) in the literal dg-framework see [D, Prop. 1.6.3]. We include a proof in the framework in which we work: Note that (ii) follows from (i) by taking $\mathcal{A} = \mathrm{Perf}(R)$ and taking into account the equivalence $\mathbf{dgcat}^{\mathrm{idm}}_R = (\mathrm{Perf}(R))\text{-mod}(\mathbf{dgcat}^{\mathrm{idm}}_k)$. To prove (i), note that it suffices to pass to the following Ind-completed version: Observe that

$$\mathrm{Ind}\,\mathcal{C}'' \longleftarrow \mathrm{Ind}\,\mathcal{C} \longleftarrow \mathrm{Ind}\,\mathcal{C}'$$

is a diagram in $(\operatorname{Ind}\mathcal{A})\text{-mod}(\mathbf{dgcat}_k^\infty)$ whose underlying diagram in Pr^L is a cofiber sequence along colimit and compact preserving maps (i.e., a recollement sequence). It suffices to show that applying $-\widehat{\otimes}_{\operatorname{Ind}\mathcal{A}} \operatorname{Ind}\mathcal{D}$ sends this to another cofiber sequence in Pr^L: All three terms are compactly generated, and the arrows are compact and colimit preserving, so that the diagram of compact objects will be a cofiber sequence of idempotent complete ∞-categories (i.e., a Drinfeld-Verdier sequence).

It thus suffices to show that $-\widehat{\otimes}_{\operatorname{Ind}\mathcal{A}} \operatorname{Ind}\mathcal{D}$ preserves the property of being a colimit diagram in Pr^L, or passing to right adjoints that it preserve the property of being a limit diagram in Pr^R. Note that the forgetful functors $(\operatorname{Ind}\mathcal{A})\text{-mod}(\mathbf{dgcat}_k^\infty) \to \mathrm{Pr}^R \to \mathbf{Cat}_\infty$ create limits, so that it is enough to show that $\operatorname{Ind}\mathcal{D}$ is dualizable over $\operatorname{Ind}\mathcal{A}$, since $\mathcal{A}^\otimes$ is rigid, so that tensoring by it preserves limits.

The hypothesis that $\mathcal{A}^\otimes$ be *symmetric*-monoidal is un-necessary. Suppose that $\mathcal{D}$ is a left $\mathcal{A}^\otimes$-module category. Since $\mathcal{A}$ was assumed rigid, one can show that $\mathcal{D}^{\mathrm{op}}$ may be equipped with the structure of right $\mathcal{A}^\otimes$-module category, heuristically given by $d \overset{\mathcal{D}^{\mathrm{op}}}{\otimes} V \overset{\mathrm{def}}{=} V^\vee \overset{\mathcal{D}}{\otimes} d$. Then, $\operatorname{Ind}(\mathcal{D}^{\mathrm{op}})$ is the $\operatorname{Ind}(\mathcal{A})$-linear dual of $\operatorname{Ind}(\mathcal{D})$, i.e., there is a natural equivalence

$$T \otimes_{\operatorname{Ind} A} \operatorname{Ind}(\mathcal{D}) \xrightarrow{\sim} \operatorname{Fun}^{ex}_{\text{mod-}\mathcal{A}}(\mathcal{D}^{\mathrm{op}}, T) = \operatorname{Fun}^{L}_{\text{mod-}\operatorname{Ind}\mathcal{A}}(\operatorname{Ind}\mathcal{D}^{\mathrm{op}}, T)$$

for $T \in \text{mod-}\operatorname{Ind}(\mathcal{A})$. □

Remark 3.1.4.3. If M is not assumed regular, then the above proof shows the following: The natural map

$$\mathrm{PreMF}(M,f) \longrightarrow \mathrm{MF}(M,f) = \mathrm{PreMF}(M,f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$$

identifies with the (idempotent completed) Drinfeld-Verdier quotient by the thick subcategory of $\mathrm{DCoh}(M_0)$ generated by the essential image of $i^*\colon \mathrm{DCoh}(M) \to \mathrm{DCoh}(M_0)$. In other words, $\mathrm{MF}(M,f)$ is modelled by Positselski's *relative category of singularities* [P1].

The above proof also showed:

Corollary 3.1.4.4. *Suppose (M,f) is an LG pair. Then, the Drinfeld-Verdier quotient sequence*

$$\mathrm{Perf}(M_0) \to \mathrm{DCoh}(M_0) \to \mathrm{DSing}(M_0)$$

is obtained, by tensoring $\mathrm{DCoh}(M_0) \otimes_{\mathrm{DCoh}(\mathbb{B})} -$, from the universal example of $M = pt$ (so that $M_0 = \mathbb{B}$).

3.1.4.5. Combining the previous Proposition with Orlov's Theorem relating *actual* matrix factorizations to DSing [O2, O5], one finally sees that the notation $\mathrm{MF}(M,f)$ is justified. Strictly speaking, Orlov's Theorem is only stated in the case where f is flat (i.e., not zero on any component). However, it is possible to show that the above definition in fact coincides with (any reasonable definition of) matrix factorizations in general:

The assignment $U \mapsto \mathrm{MF}^\infty(U,f)$ is an étale sheaf of $k(\!(\beta)\!)$-linear ∞-categories on M (Prop. A.1.3.1). The same should be true in any other reasonable definition of infinite rank matrix factorizations, so that we are reduced to the affine case. Passing to a connected component of M, we may suppose that f is either flat (covered by Orlov's Theorem), or $f = 0$ (covered by direct inspection: both categories simply give 2-periodic $\mathcal{O}_M$-modules).

3.2 Thom-Sebastiani & duality Theorems for (pre-) matrix factorizations

3.2.1 Thom-Sebastiani

3.2.1.1. For the duration of this section, suppose (M, f) and (N, g) are two LG pairs. Set

$$(M \times N)_0 = (f \boxplus g)^{-1}(0) \qquad [(M \times N)_0]_0 = (f \boxminus g)^{-1}(0)$$

Define

$$\mathrm{PreMF}(M \times N, f, g) \overset{\mathrm{def}}{=} \mathrm{DCoh}\left(M_0 \times N_0\right) \quad \text{equipped with its } k[\![\beta_M, \beta_N]\!]\text{-linear structure}$$

$$\mathrm{PreMF}(M\times N, f\boxplus g, f\boxminus g) \overset{\mathrm{def}}{=} \mathrm{DCoh}\left([M \times N]_0\right) \quad \text{equipped with its } k[\![\beta_+, \beta_-]\!]\text{-linear structure}$$

By Theorem A.2.2.4, exterior product induces a $k[\![\beta_M, \beta_N]\!]$-linear equivalence $\mathrm{PreMF}(M \times N, f, g) \simeq \mathrm{PreMF}(M, f) \otimes_k \mathrm{PreMF}(N, g)$. These two constructions are related as follows:

Lemma 3.2.1.2. *Suppose (M, f), (N, g) are two LG pairs and $Z_M \subset M$, $Z_N \subset N$ closed subsets. Then, the k-linear equivalence $\boxtimes : \mathrm{DCoh}_{Z_M}(M_0)\otimes\mathrm{DCoh}_{Z_N}(N_0) \to \mathrm{DCoh}_{Z_M\times Z_N}(M_0\times N_0)$ of Theorem A.2.2.4 naturally lifts to a*

$$k[\![\beta_+, \beta_-]\!] \xrightarrow{\sim} k[\![\beta_M, \beta_N]\!] \qquad \beta_+ \mapsto \beta_M + \beta_N, \quad \beta_- \mapsto \beta_M - \beta_N \in k[\![\beta_M, \beta_N]\!]$$

linear equivalence

$$\boxtimes\colon\ \mathrm{PreMF}_{Z_M}(M, f) \otimes \mathrm{PreMF}_{Z_N}(N, g) \longrightarrow \mathrm{PreMF}_{Z_M\times Z_N}\left(M \times N, f \boxplus g, f \boxminus g\right)$$

Proof. Set $[(M \times N)_0]_0 = (f \boxminus g)^{-1}\left((M \times N)_0\right)$. The equivalence

$$M_0 \times N_0 \xrightarrow{\sim} [(M \times N)_0]_0$$

$$(m, n, [h_f\colon f(m) \to 0], [h_g\colon g(n) \to 0]) \longmapsto (m, n, [h_f + h_g\colon f(m) + g(n) \to 0], [h_f - h_g\colon f(m) - g(n) \to 0])$$

is equivariant with respect to the following group automorphism of $\mathbb{B}^2$:

$$\left(\Delta, \overline{\Delta}\right) : \mathbb{B}^2 \longrightarrow \mathbb{B}^2 \qquad \left([h_1\colon 0 \to 0]\,, [h_2\colon 0 \to 0]\right) \longmapsto \left([h_1 + h_2\colon 0 \to 0]\,, [h_1 - h_2\colon 0 \to 0]\right)$$

The $k[\![\beta_M, \beta_N]\!]$-action on $\mathrm{DCoh}(M_0) \otimes \mathrm{DCoh}(N_0) \simeq \mathrm{DCoh}(M_0 \times N_0)$ is obtained from the above action of $\mathbb{B}^2$ on $M_0 \times N_0$. The $k[\![\beta_+, \beta_-]\!]$-action on $\mathrm{DCoh}((M \times N)_0)$ is obtained from the above action of $\mathbb{B}^2$ on $[(M \times N)_0]_0$.

It thus suffices to verify that pushforward along the group automorphism $(\Delta, \overline{\Delta})_*$ induces the indicated automorphism on the endomorphisms of the symmetric monoidal unit: In terms of the explicit model of Prop. 3.1.1.4, $(\Delta, \overline{\Delta})$ corresponds to the algebra homomorphism

$$\phi\colon \mathcal{O}_{\mathbb{B}}^{\otimes 2} \simeq k[x_+, x_-][\epsilon_i^+, \epsilon_i^-] \longrightarrow \mathcal{O}_{\mathbb{B}}^{\otimes 2} \simeq k[x_M, x_N][\epsilon_i^M, \epsilon_i^N]$$

$$\phi(x_+) = x_M + x_N \qquad \phi(\epsilon_i^+) = \epsilon_i^M + \epsilon_i^N$$
$$\phi(x_-) = x_M - x_N \qquad \phi(\epsilon_i^-) = \epsilon_i^M - \epsilon_i^N$$

and is strictly compatible with the Hopf-algebra structure maps. Consider the Koszul-Tate resolutions that give the identifications with $k[\![\beta_M, \beta_N]\!]$ and $k[\![\beta_+, \beta_-]\!]$:

$$k_{+,-} \sim k[x_+, x_-][\epsilon_i^+, \epsilon_i^-][u_+^m/m!, u_-^m/m!] \qquad \beta_+ = \frac{\partial}{\partial u_+}, \beta_- = \frac{\partial}{\partial u_-}$$

$$k_{M,N} \sim k[x_M, x_N][\epsilon_i^M, \epsilon_i^N][u_M^m/m!, u_N^m/m!] \qquad \beta_M = \frac{\partial}{\partial u_M}, \beta_N = \frac{\partial}{\partial u_N}$$

There is an isomorphism $\phi' \colon k_{+,-} \xrightarrow{\sim} \phi_* k_{M,N}$ of $\mathcal{O}_{\mathbb{B}}^{\otimes 2}$-modules: Explicitly it is the algebra map given by ϕ on the x and ϵ variables and

$$\phi'(u_+) = u_M + u_N \qquad \phi'(u_-) = u_M - u_N$$

It is now a simple check that this identifies the actions $\beta_+ = \beta_M + \beta_N$ and $\beta_- = \beta_M - \beta_N$. □

The first Main Theorem is the following result reminiscent of a Thom-Sebastiani theorem:

Theorem 3.2.1.3 (Thom-Sebastiani for Matrix Factorizations)**.** *Suppose (M, f), (N, g) are LG pairs, $(M \times N, f \boxplus g)$ their Thom-Sebastiani sum. Suppose $Z_M \subset f^{-1}(0)$ and $Z_N \subset g^{-1}(0)$ are closed subsets. (The special case $Z_M = f^{-1}(0)$, $Z_N = g^{-1}(0)$ is the main one of interest. Note that the support conditions will still matter on the product since $(f \boxplus g)^{-1}(0)$ will generally properly contain $Z_M \times Z_N$!) Then,*

(i) The external tensor product determines an equivalence *of $k[\![\beta]\!]$-linear ∞-categories*

$$\ell_*(- \boxtimes -) \colon \operatorname{PreMF}_{Z_M}(M, f) \otimes_{k[\![\beta]\!]} \operatorname{PreMF}_{Z_N}(N, g) \xrightarrow{\sim} \operatorname{PreMF}_{Z_M \times Z_N}(M \times N, f \boxplus g).$$

Passing to Ind-*completions, it induces an* equivalence *of cocomplete $k[\![\beta]\!]$-linear ∞-categories*

$$\ell_*(- \boxtimes -) \colon \operatorname{PreMF}^\infty_{Z_M}(M, f) \widehat{\otimes}_{k[\![\beta]\!]} \operatorname{PreMF}^\infty_{Z_N}(N, g) \xrightarrow{\boxtimes} \operatorname{PreMF}^\infty_{Z_M \times Z_N}(M \times N, f \boxplus g).$$

(ii) The external tensor product determines an equivalence *of $k(\!(\beta)\!)$-linear ∞-categories*

$$\ell_*(- \boxtimes -) \colon \operatorname{MF}_{Z_M}(M, f) \otimes_{k(\!(\beta)\!)} \operatorname{MF}_{Z_N}(N, g) \xrightarrow{\sim} \operatorname{MF}_{Z_M \times Z_N}(X \times Y, f \boxplus g).$$

Passing to Ind-*completions, it induces an* equivalence *of cocomplete $k(\!(\beta)\!)$-linear ∞-categories*

$$\ell_*(- \boxtimes -) \colon \operatorname{MF}^\infty_{Z_M}(X, f) \widehat{\otimes}_{k(\!(\beta)\!)} \operatorname{MF}^\infty_{Z_N}(N, g) \xrightarrow{\sim} \operatorname{MF}^\infty_{Z_M \times Z_N}(M \times N, f \boxplus g).$$

(iii) The functors of (ii) induce a $k(\!(\beta)\!)$-linear equivalence

$$\bigoplus_{\lambda \in -\operatorname{cval}(f) \cap \operatorname{cval}(g)} \operatorname{MF}(M, f + \lambda) \otimes_{k(\!(\beta)\!)} \operatorname{MF}(N, g - \lambda) \xrightarrow{\sim} \operatorname{MF}(M \times N, f \boxplus g).$$

Passing to Ind-*completions, it induces an equivalence of cocomplete $k(\!(\beta)\!)$-linear ∞-categories*

$$\bigoplus_{\lambda \in -\operatorname{cval}(f) \cap \operatorname{cval}(g)} \operatorname{MF}^\infty(M, f + \lambda) \otimes_{k(\!(\beta)\!)} \operatorname{MF}^\infty(N, g - \lambda) \xrightarrow{\sim} \operatorname{MF}^\infty(M \times N, f \boxplus g).$$

Proof. Certainly the Ind-complete versions follow from the small versions, so we will show those.

(i) Let $i\colon M_0 \to M$, $j\colon N_0 \to N$, $k\colon (M\times N)_0 \to M\times N$, and $\ell\colon M_0 \times N_0 \to (M\times N)_0$ be the various inclusions. The functor in question will be a refinement of the k-linear functor

$$\mathrm{DCoh}(M_0)\otimes \mathrm{DCoh}(N_0) \longrightarrow \mathrm{DCoh}_{M_0\times N_0}((M\times N)_0) \qquad \mathscr{F}\otimes\mathscr{G} \mapsto \ell_*(\mathscr{F}\boxtimes\mathscr{G})$$

Before writing down a $k[\![\beta]\!]$-linear functor, we show how to conclude from this: Once a $k[\![\beta]\!]$-linear functor is written down, it suffices to check that the underlying k-linear functor is an equivalence. Write

$$\mathrm{PreMF}_{Z_M}(M,f)\otimes_{k[\![\beta]\!]}\mathrm{PreMF}_{Z_N}(N,g) \overset{\boxtimes}{\simeq} \mathrm{PreMF}_{Z_M\times Z_N}(M\times N, f, g)\otimes_{k[\![\beta_M,\beta_N]\!]} k[\![\beta]\!]$$

Applying Lemma 3.2.1.2 identity this with

$$\begin{aligned} &\simeq [\mathrm{PreMF}_{Z_M\times Z_N}(M\times N, f\boxplus g, f\boxminus g)]\otimes_{k[\![\beta_+,\beta_-]\!]} k[\![\beta_+]\!] \\ &\simeq \mathrm{PreMF}_{Z_M\times Z_N}((M\times N)_0, f\boxminus g)\otimes_{k[\![\beta_-]\!]} k \end{aligned}$$

Finally, applying Cor. 3.1.2.4 we see that ℓ_* induces an equivalence

$$\overset{\ell_*}{\simeq} \mathrm{DCoh}_{Z_M\times Z_N}((M\times N)_0)$$

We now complete the proof by constructing the desired $(\mathrm{DCoh}(\mathbb{B}),\circ)$-linear functor

$$\mathrm{DCoh}(M_0)\otimes_{\mathrm{DCoh}(\mathbb{B})}\mathrm{DCoh}(N_0)\to \mathrm{DCoh}((M\times N)_0)$$

using a suitable augmented simplicial diagram of derived schemes with $\mathbb{B}$-action and $\mathbb{B}$-equivariant maps

$$X_\bullet = \left\{M_0\times\mathbb{B}^{\bullet-1}\times N_0\right\} \longrightarrow (M\times N)_0 = X_{-1}$$

constructed as follows:

- Informally (i.e., at the level of π_0 of functor of points)

  $$X_{-1}(R) = (m\in M(R), n\in N(R), [h_{f+g}\colon f(m)+g(n)\to 0])$$

  and for $\ell \geq 0$

  $$X_\ell(R) = (m\in M(R), n\in N(R), [h_f\colon f(m)\to 0], [h_g\colon g(m)\to 0], [h_1\colon 0\to 0],\cdots,[h_\ell\colon 0\to 0])$$

  with the simplicial degeneracies given by the inserting the identity loop $[\mathrm{id}\colon 0\to 0]$, and the simplicial face maps given by "pointwise addition" of the appropriate loops. The augmentation is given by adding together all the loops. It is clear that this is a simplicial diagram, and that it is $\mathbb{B}$-equivariant.
- The same formulas can be made precise, either at the level of actual functors of points on **DRng**, or using an explicit cosimplicial diagram of sheaves of dg-

algebras on $M \times N$ (representing the pushforwards of the structure sheaves to $M \times N$). For the second approach, one can use the explicit models

$$X_{-1}(R) = \mathrm{Spec}_{M\times N}\, \mathcal{O}_{M\times N}\left[x_+\right]\left[\epsilon_1^+, \epsilon_2^+\right] / \left(d\epsilon_1^+ = x_+ - (f\otimes 1 + 1 \otimes f), d\epsilon_2^+ = x_+\right)$$

$$X_\ell(R) = \mathrm{Spec}_{M\times N}\, \mathcal{O}_{M\times N}\begin{bmatrix} x_M \\ x_N \\ x_1, \dots, x_\ell \end{bmatrix}\begin{bmatrix} \epsilon_1^M, \epsilon_2^M \\ \epsilon_1^N, \epsilon_2^N \\ \epsilon_1^i, \epsilon_2^i, i = 1, \dots, \ell \end{bmatrix} \Big/ \begin{pmatrix} d\epsilon_1^M = x_M - (f\otimes 1), d\epsilon_2^M = x_M \\ d\epsilon_1^N = x_N - (1 \otimes f), d\epsilon_2^N = x_N \\ d\epsilon_1^i = d\epsilon_2^i = x \end{pmatrix}$$

where the simplicial structure maps use the co-identity and co-multiplication/co-action (c.f., proof of Prop. 3.1.1.4) and the augmentation uses those together with $x_+ \mapsto x_M + x_N$ and $\epsilon_i^+ \mapsto \epsilon_i^M + \epsilon_i^N$.

This completes the construction as follows:

- Applying $(\mathrm{DCoh}(-), f_*)$ to the augmented simplicial diagram $X_\bullet$ gives an augmented simplicial diagram $\mathrm{DCoh}(X_\bullet)$ in $\mathbf{dgcat}_k$. Applying Prop. A.2.3.2, we identify the simplicial diagram with the simplicial bar construction computing $\mathrm{DCoh}(M_0) \otimes_{\mathrm{DCoh}(\mathbb{B})} \mathrm{DCoh}(N_0)$. Thus, the augmented diagram precisely encodes a functor

$$\mathrm{DCoh}(M_0) \otimes_{\mathrm{DCoh}(\mathbb{B})} \mathrm{DCoh}(N_0) \longrightarrow \mathrm{DCoh}((M\times N)_0)$$

which is in fact an enrichment of $\ell(- \boxtimes -)$ since the map $X_0 \to X_{-1}$ is precisely ℓ.

- Since $X_\bullet$ was a $\mathbb{B}$-equivariant diagram, the previous functor is naturally $\mathrm{DCoh}(\mathbb{B})$-linear.

(ii) Follows from (i) by the definition $\mathrm{MF}(M, f) = \mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$.

(iii) Let $Z = \mathrm{crit}(f \boxplus g) \cap (M \times N)_0$ be the components of the critical locus of $f \boxplus g$ lying over zero. There is a disjoint union decomposition

$$Z = \coprod_{\lambda \in -\,\mathrm{cval}(f)\cap \mathrm{cval}(g)} Z_\lambda \qquad \text{where} \qquad Z_\lambda = Z \cap \left(f^{-1}(-\lambda) \times g^{-1}(\lambda)\right).$$

By Prop. 3.2.1.6, the inclusion induces an equivalence $\mathrm{MF}_Z(M \times N, f \boxplus g) = \mathrm{MF}(M \times N, f \boxplus g)$ and the above disjoint union decomposition gives

$$\mathrm{MF}_Z(M\times M, f\boxplus g) = \bigoplus_{\lambda \in -\,\mathrm{cval}(f)\cap \mathrm{cval}(g)} \mathrm{MF}_{Z_\lambda}(M \times M, f \boxplus g)$$

Combining with (ii) completes the proof. □

Remark 3.2.1.4. Item (iii) above admits the following re-interpretation. Define $\mathrm{MF}^{\mathrm{tot}}(M, f)$ to be the sheaf of ∞-categories on $\mathbb{A}^1$, supported on $\mathrm{cval}(f)$, given heuristically by $\lambda \mapsto \mathrm{MF}(N, f - \lambda)$. Then,

$$\mathrm{MF}^{\mathrm{tot}}(M \times N, f \boxplus g) = \mathrm{MF}^{\mathrm{tot}}(M, f) * \mathrm{MF}^{\mathrm{tot}}(N, g)$$

where $*$ denotes convolution.

Remark 3.2.1.5. In case $\operatorname{cval} g = \{0\}$, item (iii) has an especially simple formulation: $\mathrm{MF}(M, f) \otimes_{k((\beta))} \mathrm{MF}(N, g) \simeq \mathrm{MF}(M \times N, f \boxplus g)$. Taking $N = \mathbb{A}^2$, $g = x^2 + y^2$, one can show that there is a $k((\beta))$-linear equivalence $\mathrm{MF}(N, g) \simeq \operatorname{Perf} k((\beta))$. So, (iii) recovers *Knörrer Periodicity* as a special case:

$$\mathrm{MF}\left(M \times \mathbb{A}^2, f \boxplus \{x^2 + y^2\}\right) = \mathrm{MF}(M, f) \otimes_{k((\beta))} \mathrm{MF}(\mathbb{A}^2, \{x^2 + y^2\}) \simeq \mathrm{MF}(M, f).$$

The following is of course well-known, e.g., from its role in [O4]. We sketch a proof for the reader's convenience:

Proposition 3.2.1.6. *(Recall that we work in the world of idempotent complete categories, and so have implicitly passed to idempotent completions!) Suppose (M, f) is an LG pair, and $Z \subset f^{-1}(0)$ a closed set containing all components of the critical locus lying over 0. Then, the $k[\![\beta]\!]$-linear inclusion $\mathrm{PreMF}_Z(M, f) \hookrightarrow \mathrm{PreMF}(M, f)$ induces a $k((\beta))$-linear* equivalence *$\mathrm{MF}_Z(M, f) \xrightarrow{\sim} \mathrm{MF}(M, f)$.*

Sketch. Set $X = f^{-1}(0)$, $U = X \setminus Z$. A $k((\beta))$-linear functor is an equivalence iff it is an equivalence when regarded as a k-linear functor, so it is enough to prove this. Since U is regular, $\operatorname{Perf}(U) = \operatorname{DCoh}(U)$ and this follows from the following diagram of k-linear idempotent complete ∞-categories

$$\begin{array}{ccccc}
\operatorname{Perf}_Z(X) & \hookrightarrow & \operatorname{DCoh}_Z(X) & \twoheadrightarrow & \operatorname{DSing}_Z(X) \simeq \mathrm{MF}_Z(X, f) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Perf}(X) & \hookrightarrow & \operatorname{DCoh}(X) & \twoheadrightarrow & \operatorname{DSing}(X) \simeq \mathrm{MF}(X, f) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Perf}(U) & \xrightarrow{\sim} & \operatorname{DCoh}(U) & \longrightarrow & \operatorname{DSing}(U) = 0
\end{array}$$

where each row and column is a Drinfeld-Verdier quotient. □

3.2.2 Duality and functor categories

Recall the following standard Lemma

Lemma 3.2.2.1.

- *Suppose $\mathcal{A}^{\otimes} \in \mathbf{CAlg}(\mathbf{dgcat}_k^{\mathrm{idm}})$ is a rigid symmetric-monoidal dg-category, and $\mathcal{C} \in \mathcal{A}\text{-mod}(\mathbf{dgcat}_k^{\mathrm{idm}})$ is an $\mathcal{A}$-module category. Then, $\operatorname{Ind} \mathcal{C} \in \operatorname{Ind} \mathcal{A}\text{-mod}(\mathbf{dgcat}_k^{\infty})$ is dualizable, with dual $\operatorname{Ind} \mathcal{C}^{\mathrm{op}}$.*

- *Suppose R is an E_∞-algebra, and $\mathcal{C} \in \mathbf{dgcat}_R^{\mathrm{idm}}$. Then, $\operatorname{Ind} \mathcal{C} = \mathrm{dgmod}_R(\mathcal{C}^{\mathrm{op}})$ is dualizable, with dual $\operatorname{Ind} \mathcal{C}^{\mathrm{op}} = \mathrm{dgmod}_R(\mathcal{C})$.*

Proof. C.f., Lemma 3.1.4.2. □

Our second Main Theorem analyzes the interplay of passage to matrix factorizations with the usual (coherent) Grothendieck duality:

Theorem 3.2.2.2 (Duality for Matrix Factorizations)**.** *Suppose (M, f) is an LG pair, $Z \subset f^{-1}(M)$ a closed subset. Then,*

(i) *The usual Grothendieck duality lifts to a $k[\![\beta]\!]$-linear anti-equivalence $\mathbb{D}(-)\colon \operatorname{PreMF}_Z(M,f)^{\mathrm{op}} \simeq \operatorname{PreMF}_Z(M,-f)$. So, $\operatorname{PreMF}_Z^\infty(M,f)$ is dualizable as cocomplete $k[\![\beta]\!]$-linear category and (the above lift of) Grothendieck duality induces a $k[\![\beta]\!]$-linear equivalence*

$$\operatorname{PreMF}_Z^\infty(M,f)^\vee \xrightarrow{\sim} \operatorname{PreMF}_Z^\infty(M,-f).$$

(ii) *The usual Grothendieck duality lifts to a $k(\!(\beta)\!)$-linear anti-equivalence $\mathbb{D}(-)\colon \operatorname{MF}_Z(M,f)^{\mathrm{op}} \simeq \operatorname{MF}_Z(M,-f)$. So, $\operatorname{MF}_Z^\infty(M,f)$ is dualizable as cocomplete $k(\!(\beta)\!)$-linear category, and the usual Grothendieck duality functor induces an equivalence*

$$\operatorname{MF}_Z^\infty(M,f)^\vee \xrightarrow{\sim} \operatorname{MF}_Z^\infty(M,-f).$$

Proof. Note that (i) implies (ii). By Lemma 3.2.2.1, it suffices to prove either the first or the second sentence of (i): passing to compact objects, or to Ind-completions, goes between the two versions. Note that Grothendieck duality preserves support conditions: $\operatorname{DCoh}_Z(M)$ is generated by pushforwards from Z, and Grothendieck duality commutes with proper pushforward. So, it suffices to prove the version without support conditions.

In a rigid-enough dg-model for the $k[\![\beta]\!]$-linear ∞-category $\operatorname{PreMF}(M,f)$, it should be possible to prove the first sentence of (i) by direct computation. Since we are not in such a framework, we adopt a more indirect approach. It suffices to write down a colimit preserving, $k[\![\beta]\!]$-linear functor

$$\langle\cdot\rangle\colon \operatorname{PreMF}^\infty(M,-f)\widehat{\otimes}_{\mathrm{QC}^!(\mathbb{B})} \operatorname{PreMF}^\infty(M,f) \longrightarrow \mathrm{QC}^!(\mathbb{B})$$

and then show that it is a "perfect pairing" in the sense that the induced functor

$$\operatorname{PreMF}^\infty(M,-f) \to \operatorname{Fun}^L_{\mathrm{QC}^!(\mathbb{B})}\left(\operatorname{PreMF}^\infty(M,f), \mathrm{QC}^!(\mathbb{B})\right) \simeq \operatorname{Ind}\operatorname{PreMF}(M,f)^{\mathrm{op}}$$

is an equivalence.

Let $(M^2)_0 = (-f \boxplus f)^{-1}(0)$; $\ell\colon (M_0)^2 \to (M^2)_0$ and $k\colon (M^2)_0 \to M^2$ the inclusions; $\Delta\colon M \to M^2$ the diagonal, and $\overline{\Delta}\colon M \to (M^2)_0$ its factorization through k. To define $\langle\cdot\rangle$, we apply Theorem 3.2.1.3 to identify

$$\ell_*(- \boxtimes -)\colon \operatorname{PreMF}^\infty(M,-f)\widehat{\otimes}_{\mathrm{QC}^!(\mathbb{B})} \operatorname{PreMF}^\infty(M,f) \xrightarrow{\sim} \operatorname{PreMF}^\infty(M^2, -f \boxplus f)$$

and then the functor

$$\operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{PreMF}^\infty(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, -\right)\colon \operatorname{PreMF}^\infty(M^2,-f\boxplus f) \to \mathrm{QC}^!(\mathbb{B})$$

is colimit preserving (since $\overline{\Delta}_*\mathcal{O}_M \in \operatorname{DCoh}((M^2)_0)$ is compact) and naturally admits a $\mathrm{QC}^!(\mathbb{B})$-linear structure (since $\mathrm{QC}^!(\mathbb{B})$ is symmetric monoidal). Define $\langle\cdot\rangle$ as the composite

$$\langle - \otimes - \rangle = \operatorname{RHom}^{\otimes k[\![\beta]\!]}_{\operatorname{PreMF}^\infty(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \ell_*(- \boxtimes -)\right)$$

It remains to show that this induces a perfect pairing, i.e., that the adjoint map is an equivalence. For this it suffices to check that the underlying k-linear functor of the adjoint is an equivalence. But this underlying k-linear functor is simply Ind of regular Grothendieck duality for $\mathrm{DCoh}(M_0)$: Note that there is a Cartesian diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{i} & M \\ {\scriptstyle \Delta_{M_0}}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\ (M_0)^2 & \xrightarrow[\ell]{} & (M^2)_0 \end{array}$$

with i and ℓ of finite-Tor dimension. So, for $\mathscr{F}, \mathscr{G} \in \mathrm{DCoh}(M_0)$ there are natural equivalences

$$\begin{aligned} \mathrm{RHom}_{M_0}(\mathbb{D}\mathscr{F}, \mathscr{G}) &= \mathrm{R}\Gamma(M_0, \mathscr{F} \overset{!}{\otimes} \mathscr{G}) \\ &= \mathrm{RHom}_{M_0}((\Delta_{M_0})_* \mathcal{O}_{M_0}, \mathscr{F} \boxtimes \mathscr{G}) \\ &= \mathrm{RHom}_{M_0}((\Delta_{M_0})_* i^* \mathcal{O}_M, \mathscr{F} \boxtimes \mathscr{G}) \\ &= \mathrm{RHom}_{M_0}(\ell^* \overline{\Delta}_* \mathcal{O}_M, \mathscr{F} \boxtimes \mathscr{G}) \\ &= \mathrm{RHom}_{M_0}(\overline{\Delta}_* \mathcal{O}_M, \ell_*(\mathscr{F} \boxtimes \mathscr{G})) \end{aligned}$$

□

Formally combining the above two theorems, we obtain the following descriptions of functor categories:

Theorem 3.2.2.3 (Functors between Matrix Factorizations)**.** *Suppose (M, f), (N, g) are LG pairs. Then,*

(i) There is a $k[\![\beta]\!]$-linear equivalence

$$\mathrm{Fun}^L_{k[\![\beta]\!]}\left(\mathrm{PreMF}^\infty_{Z_M}(M, f), \mathrm{PreMF}^\infty_{Z_N}(N, g)\right) \xrightarrow{\sim} \mathrm{PreMF}^\infty_{Z_M \times Z_N}(M \times N, -f \boxplus g)$$

(ii) There is a $k(\!(\beta)\!)$-linear equivalence

$$\mathrm{Fun}^L_{k(\!(\beta)\!)}\left(\mathrm{MF}^\infty_{Z_M}(M, f), \mathrm{MF}^\infty_{Z_N}(N, g)\right) \xrightarrow{\sim} \mathrm{MF}^\infty_{Z_M \times Z_N}(M \times N, -f \boxplus g)$$

(iii) Summing (ii) over support conditions giving different components of $(-f \boxplus g)^{-1}(0)$ yields an equivalence

$$\bigoplus_{\lambda \in \mathrm{cval}(f) \cap \mathrm{cval}(g)} \mathrm{Fun}^L_{k(\!(\beta)\!)}(\mathrm{MF}^\infty(M, f - \lambda), \mathrm{MF}^\infty(N, g - \lambda)) \xrightarrow{\sim} \mathrm{MF}^\infty(M \times N, -f \boxplus g)$$

(iv) Specializing (i) and (ii) to the case $M = N$, $f = g$, we obtain equivalences

$$\mathrm{Fun}^L_{k[\![\beta]\!]}(\mathrm{PreMF}^\infty_Z(M, f), \mathrm{PreMF}^\infty_Z(M, f)) \xrightarrow{\sim} \mathrm{PreMF}^\infty_{Z \times Z}(M \times M, -f \boxplus f)$$

$$\mathrm{Fun}^L_{k(\!(\beta)\!)}(\mathrm{MF}^\infty_Z(M, f), \mathrm{MF}^\infty_Z(M, f)) \xrightarrow{\sim} \mathrm{MF}^\infty_{Z \times Z}(M \times M, -f \boxplus f)$$

Let $(M^2)_0 = (-f \boxplus f)^{-1}(0)$. The diagonal $\Delta \colon M \to M^2$ factors through $\overline{\Delta} \colon M \to (M^2)_0$. Set

$$\overline{\mathcal{O}_\Delta} = \overline{\Delta}_* \mathcal{O}_M \in \mathrm{DCoh}((M^2)_0), \qquad \overline{\omega_{\Delta,Z}} = \overline{\Delta}_* \, \mathrm{R}\Gamma_Z \, \omega_M \in \mathrm{Ind}\,\mathrm{DCoh}_{Z^2}((M^2)_0).$$

Under the equivalence above,

$$\mathrm{id}_{\mathrm{PreMF}^\infty_Z(M,f)} \longmapsto \overline{\omega_{\Delta,Z}} \qquad \mathrm{ev}_{\mathrm{PreMF}^\infty_Z(M,f)}(-) \longmapsto \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^\infty(M^2, -f \boxplus f)}(\overline{\mathcal{O}_\Delta}, -)$$

$$\mathrm{id}_{\mathrm{MF}^\infty_Z(M,f)} \longmapsto \overline{\omega_{\Delta,Z}} \qquad \mathrm{ev}_{\mathrm{MF}^\infty_Z(M,f)}(-) \longmapsto \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}^\infty(M^2,-f\boxplus f)}(\overline{\mathcal{O}_\Delta},-)$$

(v) *Specializing (iii) to the case $M = N$, $f = g$, we obtain an equivalence*

$$\bigoplus_{\lambda\in\mathrm{cval}(f)} \mathrm{Fun}^L_{k(\!(\beta)\!)}\left(\mathrm{MF}^\infty(M, f-\lambda), \mathrm{MF}^\infty(M, f-\lambda)\right) \xrightarrow{\sim} \mathrm{MF}^\infty(M^2, -f\boxplus f)$$

under which

$$\oplus_{\lambda\in\mathrm{cval}(f)} \mathrm{id}_{\mathrm{MF}^\infty(M,f-\lambda)} \longmapsto \overline{\omega_\Delta} = \overline{\Delta}_*\omega_M$$

$$\oplus\, \mathrm{ev}_{\mathrm{MF}^\infty(M,f-\lambda)}(-) \longmapsto \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}^\infty(M^2,-f\boxplus f)}(\overline{\mathcal{O}_\Delta},-)$$

Proof.

(i) The first equality follows from the adjunction of Fun^L_R and $\widehat{\otimes}_R$ on $\mathbf{dgcat}^\infty_R$, together with Theorem 3.2.2.2. The second from Theorem 3.2.1.3.

(ii) Base change of (i).

(iii) Combine Theorem 3.2.2.2 with the adjunction and Theorem 3.2.1.3(iii).

(iv) The only new statement is the identification of the functor represented by $\overline{\omega_{\Delta,Z}}$ and the trace. The identification of ev trace follows from the proof of Theorem 3.2.2.2. To identify $\overline{\omega_{\Delta,Z}}$ with the identify functor, we must trace through the equivalence of the theorem.

Let $i\colon M_0 \to M$, $j\colon N_0 \to N$, and $k\colon (M\times N)_0 \to M\times N$ be the inclusions. For a compact object $\mathscr{K} \in \mathrm{PreMF}_{M_0\times N_0}(M\times N, -f\boxplus g)$, let $\Phi'_{\mathscr{K}}$ denote the corresponding functor. We claim that Φ' is determined by the following refinement of the statement that $j_*\circ\Phi'_{\mathscr{K}} = \Phi_{k_*\mathscr{K}}\circ i_*$ compact objects:
Claim: There is a $k[\![\beta]\!]$-linear equivalence

$$\begin{aligned}\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^\infty_{Z_N}(N,g)}(T, \Phi'_{\mathscr{K}}(T')) &= \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^\infty(M^2\times N, f\boxplus -f\boxplus g)}\left(\overline{\Delta}_*\mathcal{O}_M\boxtimes T, T'\boxtimes\mathscr{K}\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M^2\times N)}\left(\Delta_*\mathcal{O}_M\boxtimes j_*T, i_*T'\boxtimes k_*\mathscr{K}\right)^{S^1}\\ &= \mathrm{RHom}_{\mathrm{QC}^!(N)}(j_*T, \Phi^!_{k_*\mathscr{K}}(i_*T'))^{S^1}\end{aligned}$$

naturally in $T\in\mathrm{PreMF}_{Z_N}(N,g)$ and $T'\in\mathrm{PreMF}_{Z_M}(M,f)$, where $\Phi^!_{k_*\mathscr{K}}$ denotes the shriek integral transform of Theorem A.2.2.4.
Proof of Claim: Tracing through the proof and using the previous Theorems repeatedly, we see that

$$\begin{aligned}&\mathrm{RHom}_{\mathrm{PreMF}^\infty(N,g)}\left(T, \Phi'_{\mathscr{F}\otimes\mathscr{G}}(T')\right)\\ &\quad= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^2\times N,-f\boxplus f\boxplus g)}(\overline{\Delta}_*\mathcal{O}_M, \mathscr{F}\boxtimes T')\otimes_{k[\![\beta]\!]}\mathrm{RHom}_{\mathrm{PreMF}^\infty(N,g)}(T,\mathscr{G})\\ &\quad= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^2\times N,-f\boxplus f\boxplus g)}(\overline{\Delta}_*\mathcal{O}_M\boxtimes T, \mathscr{F}\boxtimes T'\boxtimes\mathscr{G})\\ &\quad= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^2\times N,f\boxplus -f\boxplus g)}(\overline{\Delta}_*\mathcal{O}_M\boxtimes T, T'\boxtimes\mathscr{F}\boxtimes\mathscr{G})\end{aligned}$$

so that, extending by colimits, we obtain

$$\begin{aligned}\mathrm{RHom}_{\mathrm{PreMF}^\infty(N,g)}\left(T, \Phi'_{\mathscr{K}}(T')\right) &= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^2\times N, f\boxplus -f\boxplus g)}\left(\overline{\Delta}_*\mathcal{O}_M \boxtimes T, T'\boxtimes \mathscr{K}\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M^2\times N)}\left(\Delta_*\mathcal{O}_M\boxtimes j_*T, i_*T'\boxtimes k_*\mathscr{K}\right)^{S^1}\end{aligned}$$

Running the analogous argument for $\mathrm{QC}^!$ and $\Phi^!$, we identify the last line with

$$= \mathrm{RHom}_{\mathrm{QC}^!(N)}\left(j_*T, \Phi^!_{k_*\mathscr{K}}(i_*T')\right)^{S^1}$$

as claimed.

We now complete the proof: Note that the property in the claim characterizes $\Phi'_{\mathscr{K}}$ up to natural equivalence: Since $\Phi'_{\mathscr{K}}$ is colimit-preserving, it is determined by its restriction to compact objects $T' \in \mathrm{PreMF}_{Z_M}(M,f)$, and since $\mathrm{PreMF}^\infty_{Z_N}(N,g)$ is compactly-generated it is determined by the above mapping functors.

First, a feasibility check: Note that

$$k_*\overline{\omega_{\Delta,Z}} = \Delta_*\,\mathrm{R}\Gamma_Z\,\omega_M$$

so that Theorem A.2.2.4 implies that $\Phi^!_{k_*\mathscr{K}} = \mathrm{R}\Gamma_Z$ which is naturally the identity on the essential image of $i_*\colon \mathrm{DCoh}_Z(M_0)\to \mathrm{DCoh}(M)$; thus, we're done up to identifying the S^1-action. This seems inconvenient in this viewpoint, so instead we take a different approach.

Consider the (simplicial diagram of) Cartesian diagrams

$$\begin{array}{ccc} M_0\times \mathbb{B}^\bullet & \xrightarrow{D_2} & M\times M_0\times\mathbb{B}^\bullet \\ {\scriptstyle D_1}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}_1} \\ M_0\times M\times \mathbb{B}^\bullet & \xrightarrow[\overline{\Delta}_2]{} & (M^3)_0\times\mathbb{B}^\bullet \end{array}$$

where the $D_i, \overline{\Delta}_i$ are the evident diagonal maps. All the arrows finite and of finite Tor-dimension. Considering $\bullet = 0$, we obtain a k-linear identification

$$\begin{aligned}\mathrm{RHom}_{\mathrm{PreMF}^\infty(M,f)}\left(T, \Phi'_{\overline{\omega_{\Delta,Z}}}(T')\right) &= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^3, f\boxplus -f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M\boxtimes T, T'\boxtimes\overline{\Delta}_*\,\mathrm{R}\Gamma_Z\,\omega_M\right)\\ &= \mathrm{RHom}_{\mathrm{PreMF}^\infty(M^3, f\boxplus -f\boxplus f)}\left(\overline{\Delta_1}_*(\mathcal{O}_M\boxtimes T), \overline{\Delta_2}_*(T'\boxtimes\overline{\Delta}_*\,\mathrm{R}\Gamma_Z\,\omega_M)\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M\times M_0)}\left(\mathcal{O}_M\boxtimes T, \overline{\Delta_1}^!\overline{\Delta_2}_*(T'\boxtimes\overline{\Delta}_*\,\mathrm{R}\Gamma_Z\,\omega_M)\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M\times M_0)}\left(\mathcal{O}_M\boxtimes T, (D_2)_*(D_1)^!(T'\boxtimes\overline{\Delta}_*\,\mathrm{R}\Gamma_Z\,\omega_M)\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M_0)}\left((D_2)^*(\mathcal{O}_M\boxtimes T), (D_1)^!(T'\boxtimes\overline{\Delta}_*\,\mathrm{R}\Gamma_Z\,\omega_M)\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M_0)}\left(\mathcal{O}_{M_0}\otimes T, T'\overset{!}{\otimes}\mathrm{R}\Gamma_Z\,\omega_{M_0}\right)\\ &= \mathrm{RHom}_{\mathrm{QC}^!(M_0)}\left(T,T'\right) = \mathrm{RHom}_{\mathrm{PreMF}^\infty(M,f)}(T,T')\end{aligned}$$

To obtain a DCoh($\mathbb{B}$)-linear identification we apply an analogous argument for all $\bullet$, obtaining natural k-linear identifications

$$\mathrm{RHom}_{\mathrm{PreMF}^{\infty}(M,f)}\left(V_1 \otimes \cdots \otimes V_\bullet \otimes T, \Phi'_{\overline{\omega_{\Delta,Z}}}(T')\right) = \mathrm{RHom}_{\mathrm{PreMF}^{\infty}(M,f)}\left(V_1 \otimes \cdots \otimes V_\bullet \otimes T, T'\right)$$

for $V_1, \ldots, V_\bullet \in \mathrm{DCoh}(\mathbb{B})$ and $T, T' \in \mathrm{PreMF}(M, f)$.

(v) Follows from (iv). □

Remark 3.2.2.4. Note that in (iv), the Hom in the formula for ev is taking place in a category *without* support conditions. Note also that, owing to the application of $\mathrm{R}\Gamma_Z$ in obtaining the identity functor, the identity functor on PreMF will *almost never* be compact—that is, PreMF is almost never smooth over $k[\![\beta]\!]$.

Remark 3.2.2.5. The above proof of (iv) is somewhat opaque, due to the attempt to isolate and minimize the use of operations on ∞-categories. The discussion of chapter 5 allows for an argument via identifying the S^1-action, based on viewing $\overline{\omega_{\Delta,Z}}$ as a lift of $\Delta_* \mathrm{R}\Gamma_Z \,\omega_M$ to S^1-invariants for a certain action on the category of endofunctors on $\mathrm{QC}^!_Z(M)$. Meanwhile, Section 4.1 contains an alternate argument based on a description of the above equivalence via shriek integral transform functors on the simplicial diagram $M_0 \times \mathbb{B}^\bullet \times M_0$.

Remark 3.2.2.6. In the previous Theorem we have written down *an* equivalence: roughly, the one corresponding to Grothendieck duality $\mathbb{D}(-) = \mathrm{RHom}(-, \omega_{M_0})$ using the dualizing complex ω_{M_0} on M_0. Working from the viewpoint of literal matrix factorizations, it seems more natural to write down a *different* equivalence: roughly the one corresponding to Grothendieck duality $\mathbb{D}'(-) = \mathrm{RHom}(-, \omega_{M_0/M})$ using the (trivialized by f, in degree -1) relative dualizing complex $\omega_{M_0/M}$ on M_0. For instance, it is this other equivalence that is written down by Lin and Pomerleano in [LP].

3.2.2.7. *Warning:* The two equivalences give rise to *different* explicit identification of the trace and identity functors.

Chapter 4

Support and completion in DAG, and more matrix factorizations via derived (based) loops

4.1 Completion via derived Cech nerve and derived groups

In this section, we put the constructions of Section 3.1 and Section 3.2 into a more general context and use this to give what we feel are better statements and proofs. Unfortunately, making precise some parts of this requires more $(\infty, 2)$-categorical preliminaries on the relations of $f^!$ and f_* on $\mathrm{QC}^!$ than we wish to get into here. Consequently, we will only sketch these proofs (being cavalier about these compatibilities) and will further defer these sketches to their own subsection. Since we first wrote this in [P2], similar results have appeared in [GR2].

4.1.1 Motivation

The starting point for this section is the following re-interpretation of Cor. 3.1.2.4, using the identification $\mathrm{DCoh}_{M_0}(M) = \mathrm{DCoh}(\widehat{M_0})$ (c.f., Theorem 4.1.2.8):

4.1.1.1. Associated to the natural inclusion $i\colon M_0 \to \widehat{M_0} = \widehat{M_{M_0}}$, is a map $\bar{i}$ from its Cech nerve. Since $\widehat{M_0} \to M$ is a monomorphism, this identifies with

$$\widehat{M_0} \xleftarrow{\bar{i}} \left\{M_0^{\times_M \bullet}\right\} \simeq \left\{M_0 \times \mathbb{B}^{\times \bullet - 1}\right\}$$

The realization (say in étale sheaves) of the last simplicial diagram is the definition of $M_0/\mathbb{B}$, and we have constructed a map $\bar{i}\colon M_0/\mathbb{B} \to \widehat{M_{M_0}}$. At the level of R-points

- $M_0(R)$ consists of an $m \in M(R)$ together with a factorization through $f = 0$, while $\mathbb{B}(R)$ acts transitively on these factorizations. So, $(M_0/\mathbb{B})(R)$ consists of those R-points in $M(R)$ which, étale locally, admit a factorization through $f = 0$.

- Meanwhile, $\widehat{M_0}(R)$ consists of the R-points in $M(R)$ which, étale locally, admit a factorization through $f^n = 0$ for some n.

4.1.1.2. Using Theorem 4.1.2.8, identify $\mathrm{QC}^!_{M_0}(M) = \mathrm{QC}^!(M_0)$. Applying Prop. A.2.3.2

$$\begin{aligned}\mathrm{QC}^!(M_0/\mathbb{B}) &= \operatorname{holim}^{\mathrm{Pr}^R}\left\{\mathrm{QC}^!(M_0 \times \mathbb{B}^{\times \bullet -1}), f^!\right\} \\ &= \operatorname{hocolim}^{\mathrm{Pr}^L}\left\{\mathrm{QC}^!(M_0 \times \mathbb{B}^{\times \bullet -1}), f_*\right\} \\ &= \operatorname{hocolim}^{\mathrm{Pr}^L}\left\{\mathrm{QC}^!(M_0) \otimes \mathrm{QC}^!(\mathbb{B})^{\otimes \bullet -1} \otimes \mathrm{QC}^!(\mathrm{pt}), f_*\right\} \\ &= \mathrm{QC}^!(M_0)\widehat{\otimes}_{\mathrm{QC}^!(\mathbb{B})}\mathrm{QC}^!(\mathrm{pt})\end{aligned}$$

So, Cor. 3.1.2.4 may be re-interpreted as saying that the inclusion $\bar{i}$ induces an equivalence on $\mathrm{QC}^!$. The approach of this section will be to give a direct proof of this sort of statement.

4.1.2 Support conditions, completion, and (derived) Cech nerves

4.1.2.1. Recall our notation $\mathrm{QC}^!(X) \stackrel{\text{def}}{=} \operatorname{Ind}\mathrm{DCoh}(X)$; the notation is suggested by the fact that for a morphism $f\colon X \to Y$ the natural notion of pullback $\operatorname{Ind}\mathrm{DCoh}(X) \to \operatorname{Ind}\mathrm{DCoh}(Y)$ does not come from the pullback f^* of quasi-coherent complexes, but from the *shriek pullback* $f^!$ of Grothendieck duality theory. This tells us that $\mathrm{QC}^!(\widehat{X_Z})$ must be defined as, roughly, a sequence of sheaves on nilthickenings of Z related by *shriek* pullback:

Definition 4.1.2.2. Suppose $\mathscr{X} \in \operatorname{Fun}(\mathbf{DRng}^{\mathrm{fp}}, \mathbf{Sp})$ is a derived space over k. Define

$$\mathrm{QC}^!(\mathscr{X}) \stackrel{\text{def}}{=} \operatorname{holim}_{\operatorname{Spec} A \to \mathscr{X}}(\operatorname{Ind}\mathrm{DCoh}(\operatorname{Spec} A), f^!)$$

A natural transformation $f\colon \mathscr{X} \to \mathscr{Y}$ of functors gives rise to a colimit-preserving functor $f^!\colon \mathrm{QC}^!(\mathscr{Y}) \to \mathrm{QC}^!(\mathscr{X})$ by restricting the test diagram along f. We will see later (Lemma 4.1.4.1) how to define a colimit-preserving $f_*\colon \mathrm{QC}^!(\mathscr{X}) \to \mathrm{QC}^!(\mathscr{Y})$ using base-change, and that $(f_*, f^!)$ is an adjoint pair if f is (representable and) finite or close to it. Section A.1 shows that this definition is sheaf for the smooth topology, and that it coincides with $\operatorname{Ind}\mathrm{DCoh}(\mathscr{X})$ for $\mathscr{X}$ a ($\star_{\mathrm{F}}$) derived DM-stack.

Definition 4.1.2.3. Suppose $\mathscr{X}$ is a derived stack and $Z \subset \mathscr{X}$ is the complement of an open substack. Define $\widehat{\mathscr{X}_Z}$ to be the sub-functor of $\mathscr{X}$ given by

$$\widehat{\mathscr{X}_Z}(R) = \left\{t \in \mathscr{X}(R)\colon t \text{ factors set-theoretically through } Z, \text{ i.e., } \operatorname{Spec}(\pi_0 R) = t^{-1}(Z)\right\}$$

4.1.2.4. Suppose $\mathscr{X}$ is a locally Noetherian discrete stack, and $\mathcal{I}_{\mathscr{Z}}$ a defining ideal for Z. Then, the above definition agrees with the usual one when restricted to discrete R: Note that $\operatorname{Spec}_{\mathscr{X}} \mathcal{O}_{\mathscr{X}}/\mathcal{I}^n_{\mathscr{Z}} \to \mathscr{X}$ is a monomorphism on discrete rings, with

$$(\operatorname{Spec}_{\mathscr{X}} \mathcal{O}_{\mathscr{X}}/\mathcal{I}^n_{\mathscr{Z}})(R) = \{t \in \mathscr{X}(R)\colon \mathcal{I}^n_{\mathscr{Z}} \cdot R = 0\}\,, \text{and}$$

$$\left(\varinjlim \operatorname{Spec}_{\mathscr{X}} \mathcal{O}_{\mathscr{X}}/\mathcal{I}^n_Z\right)(R) = \{t \in \mathscr{X}(R)\colon \mathcal{I}_{\mathscr{Z}} \cdot R \text{ is nilpotent on } \operatorname{Spec} R\}$$

Since $\mathscr{X}$ was locally Noetherian, $\mathcal{I}_{\mathscr{Z}}$ is coherent so that $\mathcal{I}_{\mathscr{Z}} \cdot R$ is nilpotent iff it is contained in the nilradical of R (i.e., t factors set-theoretically through Z).

4.1.2.5. In the derived setting, a similar directed colimit description is possible *locally* (e.g., when there is an ample family of line bundles) using suitable Koszul complexes (c.f., the proofs of Lemma 4.1.4.2 and Lemma 5.4.1.2). But now there is also a global way to understand completions via a Cech-nerve construction, in the style of the Adams spectral sequence:

Construction 4.1.2.6. Suppose $\mathscr{X}$ is a derived space, $Z \subset \mathscr{X}$ a closed subset (i.e., compatible family of closed subsets of $\operatorname{Spec} A$ for all $\operatorname{Spec} A \to \mathscr{X}$), and $p : \mathscr{Z} \to \mathscr{X}$ a finite map having support Z (e.g., if $\mathscr{X}$ is a Noetherian derived stack one can take the discrete stack $\mathscr{Z} = Z_{\mathrm{red}}$). Form the Cech nerve of p

$$p_\bullet \colon \left\{ \mathscr{Z}_\bullet = \mathscr{Z}^{\times_{\mathscr{X}} \bullet+1} \right\} \longrightarrow X$$

Note that $|p_\bullet|$ factors through the monomorphism $i : \widehat{\mathscr{X}_Z} \to \mathscr{X}$, and let $\overline{p} : \{\mathscr{Z}_\bullet\} \to \widehat{\mathscr{X}_Z}$ be the factorization.

Note that the structure maps in this augmented simplicial diagram are all finite. Note that even if $\mathscr{Z}$ and $\mathscr{X}$ were discrete, the other terms in the Cech nerve will generally not be. As defined above, $\mathrm{QC}^!$ takes colimits of derived spaces to limits of categories so that

$$\mathrm{QC}^!\left(|\mathscr{Z}_\bullet|\right) \simeq \operatorname{Tot}\left\{\mathrm{QC}^!(\mathscr{Z}_\bullet), i^!\right\} = \operatorname{Tot}\left\{ \mathrm{QC}^!(\mathscr{Z}) \underset{(p_2)^!}{\overset{(p_1)^!}{\rightrightarrows}} \mathrm{QC}^!(\mathscr{Z} \times_X \mathscr{Z}) \cdots \right\}$$

Theorem 4.1.2.7. *With notation as in Construction 4.1.2.6, there are adjoint pairs*

$$\overline{p}_* \colon \mathrm{QC}^!\left(|\mathscr{Z}_\bullet|\right) \rightleftarrows \mathrm{QC}^!(\widehat{\mathscr{X}_Z}) \colon \overline{p}^!$$

$$|p_\bullet|_* \colon \mathrm{QC}^!\left(|\mathscr{Z}_\bullet|\right) \rightleftarrows \mathrm{QC}^!(\mathscr{X}) \colon |p_\bullet|^!$$

such that

(i) The adjoint pair $\left(\overline{p}_*, \overline{p}^!\right)$ consists of mutually inverse equivalences $\mathrm{QC}^!(|\mathscr{Z}_\bullet|) \simeq \mathrm{QC}^!(\widehat{\mathscr{X}_Z})$.

(ii) The adjoint pair $\left(|p_\bullet|_*, |p_\bullet|^!\right)$ identifies $\mathrm{QC}^!(|\mathscr{Z}_\bullet|)$ with $\mathrm{QC}^!_Z(\mathscr{X})$. More precisely, $(p_\bullet)_*$ is fully faithful with essential image $\mathrm{QC}^!_Z(\mathscr{X})$ and $(p_\bullet)_*(p_\bullet)^! \simeq \underline{\mathrm{R}\Gamma}_Z$.

As a consequence, we obtain the following result which is morally important for us:

Theorem 4.1.2.8. *Suppose that $\mathscr{X}$ is a coherent derived stack, that Z is the complement of a quasi-compact open substack, and let $i \colon \widehat{\mathscr{X}_Z} \to \mathscr{X}$ be the inclusion. With the above definition, $i^! \colon \mathrm{QC}^!(X) \to \mathrm{QC}^!(\widehat{\mathscr{X}_Z})$ restricts to an* equivalence *$i^! \colon \operatorname{Ind}\mathrm{DCoh}_Z(\mathscr{X}) = \mathrm{QC}^!_Z(\mathscr{X}) \to \mathrm{QC}^!(\widehat{\mathscr{X}_Z})$ with inverse i_*. This equivalence identifies the adjoint pairs $((i_Z)_*, \mathrm{R}\Gamma_Z)$ and $(i_*, i^!)$:*

$$\begin{array}{ccc} \mathrm{QC}^!_Z(\mathscr{X}) & \underset{\mathrm{R}\Gamma_Z}{\overset{(i_Z)_*}{\rightleftarrows}} & \mathrm{QC}^!(\mathscr{X}) \\ \sim \updownarrow & & \| \\ \mathrm{QC}^!(\widehat{\mathscr{X}_Z}) & \underset{i^!}{\overset{i_*}{\rightleftarrows}} & \mathrm{QC}^!(\mathscr{X}) \end{array}$$

(In case $\mathscr{X}$ is smooth, it turns out that $i^* \colon \mathrm{QC}(\widehat{\mathscr{X}_Z}) \to \mathrm{QC}_Z(\mathscr{X})$ is also an equivalence. However, its inverse is substantially more complicated than i_*.)

Sketch. The only thing new beyond Theorem 4.1.2.7 is that $\mathrm{QC}^!_Z(\mathscr{X}) = \operatorname{Ind}\mathrm{DCoh}_Z(\mathscr{X})$, which is the content of Lemma 4.1.4.2. For moral comfort, we sketch a less derived-looking argument (independent of derived Cech nerve) in case X is a locally Noetherian discrete stack:[1] Let $\mathcal{I}_Z$ be an ideal of definition for Z, and $X_n = \operatorname{Spec}_X \mathcal{O}_X/\mathcal{I}_Z^n$ for all $n \geq 1$. Consider the diagram

$$X_1 \xrightarrow{t_1} X_2 \xrightarrow{t_2} X_3 \xrightarrow{t_3} \cdots \longrightarrow \widehat{X_Z} \xrightarrow{i} X$$

where each of the t_i is proper, so that $((t_i)_*, (t_i)^!)$ is an adjoint pair. Observe that by definition, together with the previous adjunction,

$$\begin{aligned}
\mathrm{QC}^!(\widehat{X_Z}) &= \varprojlim^{\mathbf{Cat}_\infty} \left\{ \mathrm{QC}^!(X_1) \xleftarrow{(t_1)^!} \mathrm{QC}^!(X_2) \xleftarrow{(t_2)^!} \cdots \xleftarrow{(t_{n-1})^!} \mathrm{QC}^!(X_n) \xleftarrow{(t_n)^!} \cdots \right\} \\
&= \varprojlim^{\mathrm{Pr}^R} \left\{ \mathrm{QC}^!(X_1) \xleftarrow{(t_1)^!} \mathrm{QC}^!(X_2) \xleftarrow{(t_2)^!} \cdots \xleftarrow{(t_{n-1})^!} \mathrm{QC}^!(X_n) \xleftarrow{(t_n)^!} \cdots \right\} \\
&= \varinjlim^{\mathrm{Pr}^L} \left\{ \mathrm{QC}^!(X_1) \xrightarrow{(t_1)_*} \mathrm{QC}^!(X_2) \xrightarrow{(t_2)_*} \cdots \xrightarrow{(t_{n-1})_*} \mathrm{QC}^!(X_n) \xrightarrow{(t_n)_*} \cdots \right\}
\end{aligned}$$

Since t_n is proper, the functor $(t_n)_* \colon \mathrm{QC}^!(X_n) \to \mathrm{QC}^!(X_{n+1})$ will preserve compact objects. We have the following recipe for forming a colimit in Pr^L of compactly generated categories along left-adjoints preserving compact objects: Take Ind of the colimit of categories of compact objects. In this case, this identifies the previous displayed line as

$$\begin{aligned}
&= \varinjlim^{\mathrm{Pr}^L} \left\{ \mathrm{QC}^!(X_1) \xrightarrow{(t_1)_*} \mathrm{QC}^!(X_2) \xrightarrow{(t_2)_*} \cdots \xrightarrow{(t_{n-1})_*} \mathrm{QC}^!(X_n) \xrightarrow{(t_n)_*} \cdots \right\} \\
&= \mathrm{Ind}\left(\varinjlim^{\mathbf{Cat}_\infty} \left\{ \mathrm{DCoh}(X_1) \xrightarrow{(t_1)_*} \mathrm{DCoh}(X_2) \xrightarrow{(t_2)_*} \cdots \xrightarrow{(t_{n-1})_*} \mathrm{DCoh}(X_n) \xrightarrow{(t_n)_*} \cdots \right\} \right)
\end{aligned}$$

One can show, essentially by computing local cohomology, that the natural functor $\varinjlim \mathrm{DCoh}(X_n) \to \mathrm{DCoh}_Z(X)$ is fully-faithful; it is essentially surjective by Lemma 2.2.0.2. Combining with Lemma 4.1.4.2, we identify the previous displayed line with

$$= \mathrm{Ind}(\mathrm{DCoh}_Z X) = \mathrm{QC}^!_Z(X).$$ □

Remark 4.1.2.9. Passing to compact objects, one obtains $\mathrm{DCoh}_Z(X) = \mathrm{DCoh}(\widehat{X_Z})$ where now $\mathrm{DCoh}(\widehat{X_Z}) \stackrel{\mathrm{def}}{=} \mathrm{QC}^!(\widehat{X_Z})^c$ are what one might normally call the *torsion* coherent complexes.

4.1.3 Geometric Koszul duality for $\mathrm{QC}^!$

4.1.3.1. For the duration of this section:

- We work over a base S, which is assumed to be a smooth stack over k that is *very good* in the sense that the conclusion of Prop. A.2.3.2 and Theorem A.2.2.4 holds over S (where in interpreting Theorem A.2.2.4 we must work relative to S, i.e., the "dualizing complex" of $f \colon X \to S$ is $f^!(\mathcal{O}_S)$). In particular, when we write pt we mean S.
- $\mathscr{Y}$ be (the functor of points of) a smooth formal S-scheme, $\mathrm{pt} \in \mathscr{Y}$. Let $\widehat{\mathrm{pt}}$ denote the formal completion of $\mathrm{pt} \in \mathscr{Y}$, i.e., $\operatorname{Spf} \widehat{\mathcal{O}_{\mathscr{Y},y}}$. Note that $\widehat{\mathrm{pt}}$ is also a formal scheme, and

[1] Locally on $\mathscr{X}$, a similar argument can be made in the derived setting by replacing powers of $\mathcal{I}_Z$ by a suitable filtered diagram of Koszul-type complexes.↑

the map $\widehat{\mathrm{pt}} \to \mathscr{Y}$ is an inclusion of connected components on functors of points.

– $\mathbb{G} \stackrel{\text{def}}{=} \mathrm{pt} \times_{\mathscr{Y}} \mathrm{pt}$ viewed as a derived (formal) group scheme by "composition of loops" as in §3.1.1. If $\mathscr{Y}$ is itself a (commutative) group formal scheme, then $\mathbb{G}$ may be equipped with a compatible (commutative) group structure by "pointwise addition" of loops (also as in §3.1.1). We will mostly ignore "pointwise addition" in this section, but one can put it back in to obtain $\mathrm{QC}^!(\mathbb{G})^{\otimes}$-linear statements as follows: After one writes down the relevant functors using pointwise addition (instead of loop composition), using the added $\mathbb{G}$-equivariance coming from using the commutative product, it suffices to check that the underlying k-linear functors are equivalence; an Eckmann-Hilton argument show that this k-linear functor is homotopic to that gotten by using composition of loops, thereby reducing to the case considered in this subsection.

– $\mathscr{X}, \mathscr{X}' \in \mathrm{Fun}(\mathbf{DRng}^{\mathrm{fp}}, \mathbf{Sp})$ be derived spaces, equipped with natural transformations $f\colon \mathscr{X} \to Y$ and $g\colon \mathscr{X}' \to Y$ which are relative ($\star_{\mathrm{F}}$) derived DM stacks.

Construction 4.1.3.2. Imitating §3.1.1 we observe

– $\mathscr{X}_{\mathrm{pt}} = \mathscr{X} \times_{\mathscr{Y}} \mathrm{pt}$ and $\mathscr{X}'_{\mathrm{pt}} = \mathscr{X}' \times_{\mathscr{Y}} \mathrm{pt}$ are right $\mathbb{G}$-schemes, via "composition of loops."

– $\overline{\mathscr{X}}_{\mathrm{pt}}, \overline{\mathscr{X}'}_{\mathrm{pt}}$ are the left $\mathbb{G}$-stacks obtained from $\mathscr{X}_{\mathrm{pt}}, \mathscr{X}'_{\mathrm{pt}}$ using the inverse ("read loop backwards") $i\colon \mathbb{G}^{\mathrm{op}} \simeq \mathbb{G}$.

– ${}_{\mathrm{pt}}\mathscr{X} = \mathrm{pt} \times_{\mathscr{Y}} \mathscr{X}$, and ${}_{\mathrm{pt}}\mathscr{X}' = \mathrm{pt} \times_{\mathscr{Y}} \mathscr{X}'$ are left $\mathbb{G}$-stacks.

– ${}_{\mathrm{pt}}\overline{\mathscr{X}}, {}_{\mathrm{pt}}\overline{\mathscr{X}'}$ are the right $\mathbb{G}$-stacks obtained from ${}_{\mathrm{pt}}\mathscr{X}, {}_{\mathrm{pt}}\mathscr{X}'$ using the inverse $i\colon \mathbb{G}^{\mathrm{op}} \simeq \mathbb{G}$.

– There are obvious ($\mathbb{G}$-equivariant) equivalences $\mathscr{X}_{\mathrm{pt}} \simeq {}_{\mathrm{pt}}\overline{\mathscr{X}}$, ${}_{\mathrm{pt}}\mathscr{X}' \simeq \overline{\mathscr{X}'}_{\mathrm{pt}}$, etc.

We now isolate a key part of the proofs of Theorem 3.2.1.3 and Theorem 3.2.2.3:

Construction 4.1.3.3. Consider the "Koszul duality" map of derived spaces over $\mathscr{Y}^2$:[2]

$$B\Omega_{\mathrm{pt}}\mathscr{Y} = \mathrm{pt}/\!\!/\mathbb{G} = \left| \cdots \mathbb{G}^2 \rightrightarrows \mathbb{G} \rightrightarrows \mathrm{pt} \right| \to \widehat{\mathrm{pt}}$$

Base-changing, we obtain an augmented simplicial diagram

$$\left\{ \mathscr{X} \times_{\mathscr{Y}} \mathbb{G}^{\bullet} \times_{\mathscr{Y}} \mathscr{X}' \right\} \longrightarrow \mathscr{X} \times_{\mathscr{Y}} \widehat{\mathrm{pt}} \times_{\mathscr{Y}} \mathscr{X}' \left(\hookrightarrow \mathscr{X} \times_{\mathscr{Y}} \mathscr{Y} \times_{\mathscr{Y}} \mathscr{X}' \right)$$

given heuristically on functor-of-points by

$$\begin{aligned} &\big(x \in \mathscr{X}(R), x' \in \mathscr{X}'(R), [h_f\colon f(x) \to \mathrm{pt}], [h_1\colon \mathrm{pt} \to \mathrm{pt}], \ldots, [h_\bullet\colon \mathrm{pt} \to \mathrm{pt}], [h_g\colon \mathrm{pt} \to g(x')]\big) \\ &\longmapsto \big(x \in \mathscr{X}(R), y \in \mathscr{Y}(R), [h_f \cdot h_1 \cdot \cdots \cdot h_\bullet \cdot h_g\colon f(x) \to g(x')]\big) \end{aligned}$$

Taking geometric-realization, this gives a map

$$i\colon \mathscr{X}_{\mathrm{pt}} \overset{\mathbb{G}}{\times} \overline{\mathscr{X}'}_{\mathrm{pt}} \longrightarrow \mathscr{X} \times_{\mathscr{Y}} \widehat{\mathrm{pt}} \times_{\mathscr{Y}} \mathscr{X}' = \mathscr{X}_{\mathrm{pt}} \widehat{\times}\, {}_{\mathrm{pt}}\mathscr{X}$$

[2] On R-points, $\widehat{\mathrm{pt}}(R)$ is the union of connected components of $\mathscr{Y}(R)$ consisting of maps such that, étale locally, the reduced pair $(\operatorname{Spec} \pi_0 R)_{\mathrm{red}}$ admits a factorization through $\mathrm{pt} \to Y$; $B\Omega_{\mathrm{pt}}\mathscr{Y}(R)$ is the union of connected components of $\mathscr{Y}(R)$ consisting of maps which étale locally themselves admit a factorization through $\mathrm{pt} \to \mathscr{Y}$.↑

There is the following "tensor product theorem," which roughly asserts that although $\mathrm{pt}/\!\!/\mathbb{G} \to \widehat{\mathrm{pt}}$ is not an equivalence, it is universally an equivalence on $\mathrm{QC}^!$:

Theorem 4.1.3.4. *There is a commuting diagram of equivalences*

$$\begin{array}{ccc} \mathrm{QC}^!\left(\mathscr{X}_{pt} \overset{\mathbb{G}}{\times} {}_{pt}\mathscr{X}'\right) & \underset{i^!}{\overset{i_*}{\rightleftarrows}} & \mathrm{QC}^!\left(\mathscr{X} \times_{\mathscr{Y}} \widehat{\mathrm{pt}} \times_{\mathscr{Y}} \mathscr{X}'\right) \\ \boxtimes \uparrow \sim & & i^! \uparrow \sim \downarrow i_* \\ \mathrm{QC}^!(X_{pt}) \widehat{\otimes}_{\mathrm{QC}^!(\mathbb{G})} \mathrm{QC}^!({}_{pt}\mathscr{X}') & \xrightarrow[\sim]{\boxtimes} & \mathrm{QC}^!_{\mathscr{X}_{pt} \times {}_{pt}\mathscr{X}'}(\mathscr{X} \times_{\mathscr{Y}} \mathscr{X}') \end{array}$$

Proof. It suffices to prove that the functors in the left column, right column, and top row are equivalences. The right column follows by Theorem 4.1.2.8. The top row follows from Theorem 4.1.2.7, since the simplicial object for $\mathscr{X}_{\mathrm{pt}} \overset{\mathbb{G}}{\times} {}_{\mathrm{pt}}\mathscr{X}'$ is nothing but the Cech nerve for $\mathscr{X}_{\mathrm{pt}} \times {}_{\mathrm{pt}}\mathscr{X}' \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}'$. It remains to handle the left-column: Applying $\mathrm{QC}^!$ to the simplicial diagram defining $\mathscr{X}_{\mathrm{pt}} \overset{\mathbb{G}}{\times} {}_{\mathrm{pt}}\mathscr{X}'$, and using that the structure maps are finite so that $(f_*, f^!)$ is an adjoint pair, we find

$$\begin{aligned} \mathrm{QC}^!\left(\mathscr{X}_{\mathrm{pt}} \overset{\mathbb{G}}{\times} \overline{\mathscr{X}'}_{\mathrm{pt}}\right) &= \mathrm{Tot}\left(\mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}} \times \mathbb{G}^\bullet \times {}_{\mathrm{pt}}\mathscr{X}'), f^!\right) \\ &= \left|\mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}} \times \mathbb{G}^\bullet \times {}_{\mathrm{pt}}\mathscr{X}'), f_*\right|^{\mathrm{Pr}^L} \\ &= \mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}}) \widehat{\otimes}_{\mathrm{QC}^!(\mathbb{G})} \mathrm{QC}^!({}_{\mathrm{pt}}\mathscr{X}') \end{aligned}$$

where the last equality is computing the relative tensor product by a bar-construction. □

Remark 4.1.3.5. Suitably interpreted, a version of the previous Theorem is true more generally (e.g., replacing $\mathrm{pt} \to \mathscr{Y}$ by an lci map $i\colon Z \to M$):

$$\mathrm{QC}^!\left(Z/\!\!/^Z \widehat{(Z \times_M Z)}_Z\right) \simeq \mathrm{QC}^!\left(\widehat{M_Z}\right)$$

The case of i a regular closed immersion can be deduced from the above. The case of i smooth is the equivalence of D-modules via the de Rham stack and D-modules as crystals. See [GR2] and/or [GR1].

Once this is done, we can deduce an identification of functor categories (which is perhaps more clear than Theorem 3.2.2.3; e.g., it makes identifying the identity functor more straightforward):

Theorem 4.1.3.6. *The categories of Theorem 4.1.3.4 are all equivalent to*

$$\mathrm{Fun}^L_{\mathrm{QC}^!(\mathbb{G})}\left(\mathrm{QC}^!(\mathscr{X}'_{pt}), \mathrm{QC}^!(\mathscr{X}_{pt})\right)$$

via a cosimplicial diagram of shriek integral transform functors

$${}^\bullet\Phi^!\colon\ \mathrm{QC}^!\left(\mathscr{X}_{pt} \times \mathbb{G}^\bullet \times {}_{pt}\mathscr{X}'\right) \longrightarrow \mathrm{Fun}^L_k\left(\mathrm{QC}^!(\mathscr{X}'_{pt} \times \mathbb{G}^\bullet), \mathrm{QC}^!(\mathscr{X}_{pt})\right)$$

Proof. To see this, we use Theorem A.2.2.4 and the explicit cobar resolution of the functor category:

$$\begin{aligned}\mathrm{QC}^!\left(\mathscr{X}_{\mathrm{pt}} \overset{\mathbb{G}}{\times} {}_{\mathrm{pt}}\mathscr{X}'\right) &\simeq \mathrm{Tot}\left\{\mathrm{QC}^!\left(\mathscr{X}_{\mathrm{pt}} \times \mathbb{G}^{\bullet} \times {}_{\mathrm{pt}}\mathscr{X}'\right)\right\}\\ &\xrightarrow{{}^{\bullet}\Phi^!} \mathrm{Tot}\left\{\mathrm{Fun}^L\left(\mathrm{QC}^!(\mathscr{X}'_{\mathrm{pt}} \times \mathbb{G}^{\bullet}), \mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}})\right)\right\}\\ &\xrightarrow{\mathrm{Fun}^L(\boxtimes,-)} \mathrm{Tot}\left\{\mathrm{Fun}^L\left(\mathrm{QC}^!(\mathscr{X}'_{\mathrm{pt}})\widehat{\otimes}\,\mathrm{QC}^!(\mathbb{G})^{\widehat{\otimes}\bullet}, \mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}})\right)\right\}\\ &= \mathrm{Fun}^L_{\mathrm{QC}^!(\mathbb{G})}(\mathrm{QC}^!(\mathscr{X}'_{\mathrm{pt}}), \mathrm{QC}^!(\mathscr{X}_{\mathrm{pt}}))\end{aligned}$$

We must verify that the various

$${}^{\bullet}\Phi^!\colon\ \mathrm{QC}^!\left(\mathbb{G}^{\bullet} \times X_{\mathrm{pt}} \times {}_{\mathrm{pt}}X'\right) \longrightarrow \mathrm{Fun}^L\left(\mathrm{QC}^!(\mathbb{G}^{\bullet} \times {}_{\mathrm{pt}}X), \mathrm{QC}^!({}_{\mathrm{pt}}X')\right)$$

commute with the cosimplicial structure maps. The first instance of this verification is the following: Let

$$\widetilde{\alpha}, \widetilde{\alpha}'\colon \mathbb{G} \times X_{\mathrm{pt}} \times {}_{\mathrm{pt}}X' \longrightarrow X_{\mathrm{pt}} \times {}_{\mathrm{pt}}X'$$

be given by

$$\widetilde{\alpha}(g, x, x') = (xg, x') = (g^{-1}x, x') \qquad \text{and} \qquad \widetilde{\alpha}'(g, x, x') = (x, gx') = (x, x'g^{-1})$$

Then, there are natural equivalences

$$\Phi^!_{(\widetilde{\alpha}')^!\mathscr{K}}(V \boxtimes \mathscr{F}) \simeq \Phi^!_{\mathscr{K}}(V \otimes \mathscr{F}) \qquad \text{and} \qquad \Phi^!_{\widetilde{\alpha}^!\mathscr{K}}(V \boxtimes \mathscr{F}) = V \otimes \Phi^!_{\mathscr{K}}(\mathscr{F})$$

The verification is routine using projection, base-change, etc. □

4.1.4 Sketch proofs

Lemma 4.1.4.1. *Suppose $i\colon \mathscr{X}' \to \mathscr{X}$ is a map of derived spaces. Then,*

(i) There is a well-defined functor $i_*\colon \mathrm{QC}^!(\mathscr{X}') \to \mathrm{QC}^!(\mathscr{X})$

$$(i_*\mathscr{F})(\alpha\colon \mathrm{Spec}\,R \to \mathscr{X}) \overset{def}{=} \operatorname*{hocolim}_{\begin{array}{ccc}\mathrm{Spec}\,R' & \xrightarrow{i'} & \mathrm{Spec}\,R\\ \downarrow & & \downarrow\\ \mathscr{X}' & \xrightarrow[i]{} & \mathscr{X}\end{array}} (i')_*\mathscr{F}(R' \to \mathscr{X}')$$

together with a natural map $i_*i^! \to \mathrm{id}$.

(ii) If i is a monomorphism, then there is a natural equivalence $\mathrm{id} \xrightarrow{\sim} i^!i_*$;

(iii) Suppose that $i\colon \mathscr{X}' \to \mathscr{X}$ can be written as a colimit of $i_\alpha\colon \mathscr{Z}_\alpha \to \mathscr{X}$ with each i_α and each transition map finite. Then, the map $i_*i^! \to \mathrm{id}$ of (i) is the counit of an adjunction $(i_*, i^!)$.

(iv) Suppose the hypotheses of (iii) are satisfied, that i is a monomorphism, and that $i^!$ is conservative. Then, $i^!$ and i_* are mutually inverse equivalences.

Proof.

(i) The structure maps in the hocolim defining i_* are defined as follows: Given an arrow $j\colon \operatorname{Spec} R'' \to \operatorname{Spec} R'$ over $\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}'$. There is a natural equivalence $\mathscr{F}(R'' \to \mathscr{X}') \to j^! \mathscr{F}(R' \to \mathscr{X}')$ giving rise to a composite

$$j_* \mathscr{F}(R'' \to \mathscr{X}') \to j_* j^! \mathscr{F}(R' \to \mathscr{X}') \to \mathscr{F}(R' \to \mathscr{X}')$$

where the second arrow is the candidate co-unit that always exists (though is not always a counit of an adjunction).

To prove existence of $i^* i^! \to \mathrm{id}$, consider

$$(i_* i^! \mathscr{F})(\operatorname{Spec} R \to \mathscr{X}) = \operatorname{hocolim}_{\operatorname{Spec} R' \to \operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}'} (i')_* \mathscr{F}(\operatorname{Spec} R' \to \mathscr{X}' \to \mathscr{X})$$

and compose with the natural arrow

$$(i')_* \mathscr{F}(\operatorname{Spec} R' \to \mathscr{X}' \to \mathscr{X}) \to (i')_* (i')^! \mathscr{F}(\operatorname{Spec} R \to \mathscr{X}) \to \mathscr{F}(\operatorname{Spec} R \to \mathscr{X})$$

(ii) If i is a monomorphism, then $\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}' = \mathscr{X}'$. So, for any $\operatorname{Spec} R' \to \mathscr{X}'$ the following diagram is Cartesian

$$\begin{array}{ccc} \operatorname{Spec} R' & = & \operatorname{Spec} R' \\ \downarrow & & \downarrow \\ \mathscr{X}' & = & \mathscr{X}' \\ \downarrow & & \downarrow \\ \mathscr{X}' & \longrightarrow & \mathscr{X} \end{array}$$

Writing

$$\begin{aligned} (i^! i_* \mathscr{F})(\operatorname{Spec} R' \to \mathscr{X}') &= i_*(\operatorname{Spec} R' \to \mathscr{X}' \to \mathscr{X}) \\ &= \operatorname{hocolim}_{\operatorname{Spec} R_2' \to \operatorname{Spec} R' \times_{\mathscr{X}} \mathscr{X}'} (i')_* \mathscr{F}(R_2' \to \mathscr{X}') \end{aligned}$$

we see that the diagram over which the hocolim is taken has a terminal object, given by $\operatorname{Spec} R'$ itself. The inclusion of this terminal object induces a natural equivalence $\mathscr{F}(\operatorname{Spec} R' \to \mathscr{X}') \xrightarrow{\sim} (i^! i_* \mathscr{F})(\operatorname{Spec} R' \to \mathscr{X}')$.

(iii) We first handle the case of i itself finite. Since i is affine, i_* takes on an especially nice form

$$(i_* \mathscr{F})(\operatorname{Spec} R \to \mathscr{X}) = (i')_* \mathscr{F}(\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}')$$

since $\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}'$ is again affine. Since fiber products commute with colimits in a (pre-)sheaf category, we have

$$\mathscr{X}' = \mathscr{X} \times_{\mathscr{X}} \mathscr{X}' = \operatorname{hocolim}_{\operatorname{Spec} R \to \mathscr{X}} (\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}')$$

and so $\mathrm{QC}^!(\mathscr{X}') = \lim_{\operatorname{Spec} R \to \mathscr{X}} \mathrm{QC}^!(\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}')$ and we may identify $i^!$ and i_* with the limits

$$i_*\colon\ \mathrm{QC}^!(X') = \lim_{\operatorname{Spec} R \to \mathscr{X}} \mathrm{QC}^!(\operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}') \Leftrightarrow \lim_{\operatorname{Spec} R \to \mathscr{X}} \mathrm{QC}^!(\operatorname{Spec} R) = \mathrm{QC}^!(X) \colon i^!$$

Since i is finite, so is each $i' \colon \operatorname{Spec} R \times_{\mathscr{X}} \mathscr{X}' \to \operatorname{Spec} R$. Since they are adjoint at each stage of the limit via the indicated counit, the same is true of the limit.

Now the general case: Note that we have $\mathrm{QC}^!(\mathscr{X}') = \operatorname{holim} \mathrm{QC}^!(\mathscr{Z}_\alpha)$ and $i^! = \operatorname{holim} i^!_\alpha$. Since the transition maps are finite, the above implies that we're taking a holimit of a diagram in Pr^R; also by the above, the $(i_\alpha)^!$ are morphisms in Pr^R, and so general non-sense tells us that the same is true of $i^!$. So, $i^!$ admits a left adjoint which general non-sense tells us is the colimit of the left adjoints $(i_\alpha)_*$; this colimit coincides with i_* by inspection.

(iv) If follows by (iii) that there is an adjunction $(i_*, i^!)$. It follows from (ii) that i_* is fully-faithful, and it suffices to show that it is essentially surjective. Considering the factorization of the identity $\mathrm{id}_{i_*\mathscr{F}} = \mathrm{counit}_{i_*\mathscr{F}} \circ i^!(\mathrm{unit}_{\mathscr{F}})$, we see that $i^!(\mathrm{unit}_{\mathscr{F}})$ is an equivalence for each $\mathscr{F}$. Since $i^!$ is conservative we see that the unit for the adjunction is an equivalence, completing the proof. □

Lemma 4.1.4.2. *Suppose $\mathscr{X}$ is a $(\star_{\mathrm{F}})$ derived stack, that $j\colon U \to \mathscr{X}$ is a quasi-compact open substack, and Z its closed complement. Set $\mathrm{QC}^!_Z(\mathscr{X}) = \ker j^* \colon \operatorname{Ind}\mathrm{DCoh}\, X \to \operatorname{Ind}\mathrm{DCoh}\, U$ and $\mathrm{DCoh}_Z(\mathscr{X}) = \mathrm{QC}^!_Z(\mathscr{X}) \cap \mathrm{DCoh}(\mathscr{X})$. Then, there is a natural equivalence*

$$\operatorname{Ind}\mathrm{DCoh}_Z(\mathscr{X}) = \mathrm{QC}^!_Z(\mathscr{X})$$

Sketch. If there are line bundles $\mathscr{L}_i$, $i = 1, \ldots, k$, and sections $s_i \in \Gamma(\mathscr{X}, \mathscr{L}_i)$ such that $U = \bigcup_i D(s_i)$ then one has an explicit model for $j^!$ by inverting the sections, and one can show that $\mathrm{QC}^!_Z(\mathscr{X})$ is generated by Koszul-type objects

$$\mathscr{K} \otimes \bigotimes_{i=1}^{k} \operatorname{cone}\left\{s_i^{n_i} \colon \mathscr{L}_i^{\otimes -n_i} \to \mathcal{O}\right\} \qquad \mathscr{K} \in \mathrm{DCoh}(\mathscr{X}), n_i \in \mathbb{Z}_{>0}$$

Indeed, suppose $\mathscr{F} \in \mathrm{QC}^!_Z(\mathscr{X})$, $\mathscr{K} \in \mathrm{DCoh}(\mathscr{X})$ and that $\phi\colon \mathscr{K} \to \mathscr{F}$ is a map in $\operatorname{Ind}\mathrm{DCoh}(\mathscr{X})$. The formula for j^* as a filtered colimit under multiplication by the s_i, together with compactness of $\mathscr{K}$, implies that there exist $n_i > 0$ such that $s_i^{n_i} \circ \phi$ is null-homotopic. A choice of null-homotopies then gives rise to a factorization of ϕ through the appropriate Koszul-type object of $\mathrm{DCoh}_Z(\mathscr{X})$.

When $\mathscr{X}$ is affine, or more generally has an ample family of line bundles, this is automatically satisfied. In general, $\operatorname{Ind}\mathrm{DCoh}_Z(\mathscr{X}) \to \mathrm{QC}^!_Z(\mathscr{X})$ is fully-faithful, since objects in $\mathrm{DCoh}_Z(\mathscr{X})$ are compact in $\operatorname{Ind}\mathrm{DCoh}(\mathscr{X})$ and $\mathrm{QC}^!_Z(\mathscr{X})$ is closed under colimits in $\operatorname{Ind}\mathrm{DCoh}(\mathscr{X})$. We have just proved that it is an equivalence locally, so that it suffices to verify that $\mathrm{QC}^!_Z(\mathscr{X})$ has smooth descent. Since the formation of j^* commutes with smooth base change, it suffices to note that $\mathrm{QC}^!(\mathscr{X})$ is a sheaf on $\mathscr{X}_{sm}$ (Theorem A.1.2.5). □

Sketch proof of Theorem 4.1.2.7.

(i) First observe that $\overline{p}$ is a monomorphism: It suffices to check this on R-points before étale sheafification, where it is just the claim that $|\operatorname{cosk} f\colon X \to S| \to S$ is a monomorphism for any map $f\colon X \to S$ of spaces (c.f., [L9, Prop. 6.2.3.4]). Furthermore the hypotheses of Lemma 4.1.4.1(iii) are visibly satisfied (consider the simplicial diagram). Applying Lemma 4.1.4.1(iv) it suffices to to show that $\overline{p}^!$ is conservative. Letting $\overline{p}_0 : \mathscr{Z} \to \widehat{\mathscr{X}_Z}$, it suffices to show that $\overline{p}_0^!$ is conservative.

Since $\widehat{\mathscr{X}_Z} \to \mathscr{X}$ is a monomorphism, $\overline{p}_0$ is affine (indeed finite) since $\mathscr{Z} \to \mathscr{X}$ is. Suppose $\mathscr{F} \in \mathrm{QC}^!(\widehat{\mathscr{X}_Z})$ is non-zero; then by definition, there is some $t\colon T \to \widehat{\mathscr{X}_Z} \subset \mathscr{X}$ such that $t^!\mathscr{F} \neq 0$. Let $t'\colon T' \to \mathscr{Z}$ be the base change of t along the affine morphism $\overline{p}_0$, and $p'\colon T' \to T$ the corresponding base-change of $\overline{p}_0$. Note that $(t')^!(\overline{p}_0)^!\mathscr{F} = (p')^!t^!\mathscr{F}$, so it suffices to prove that $(t')^!$ is conservative. This follows from Lemma 2.2.0.2, upon noting that $T' \to T$ is an equivalence on reduced parts since it is base-changed from $\mathscr{Z} \to \widehat{\mathscr{X}_Z}$.

(ii) The composite $|p_\bullet| = i \circ \overline{p}$ is again a monomorphism. So Lemma 4.1.4.1(iii) shows that $(|p_\bullet|)_*$ is fully-faithful and left-adjoint to $(|p_\bullet|)^!$. Let $j\colon U \to \mathscr{X}$ be the inclusion of the open complement to Z. Base-change implies $j^*(|p_\bullet|)_* = 0$ so that the essential image of $(|p_\bullet|)_*$ is contained in $\ker j^! = \mathrm{QC}^!_Z(\mathscr{X})$. It suffices to show that the restriction of $(|p_\bullet|)^!$ to $\mathrm{QC}^!_Z(\mathscr{X})$ is conservative. In light of (i), it suffices to show that the restriction $i^!|_{\mathrm{QC}^!_Z(\mathscr{X})}\colon \mathrm{QC}^!_Z(\mathscr{X}) \to \mathrm{QC}^!(\widehat{\mathscr{X}_Z})$ is conservative.

Suppose $\mathscr{F} \in \mathrm{QC}^!_Z(\mathscr{X})$ is non-zero, so that by definition there is some $t\colon T \to \mathscr{X}$ such that $t^!\mathscr{F} \neq 0$. Let $p'\colon T' = (t^{-1}(Z))_{\mathrm{red}} \to T$ be the reduced-induced (discrete) scheme structure on $t^{-1}(Z) \subset \pi_0 T$, and note that $t \circ p'\colon T' \to \mathscr{X}$ factors through $\widehat{\mathscr{X}_Z}$. Thus, it suffices to show that $(p')^!\colon \mathrm{QC}^!_{t^{-1}(Z)}(T) \to \mathrm{QC}^!(T')$ is conservative. Since $\mathrm{QC}^!_{t^{-1}(Z)}(T) = \mathrm{Ind}\,\mathrm{DCoh}_{t^{-1}(Z)}(T)$ by Lemma 4.1.4.2 in the affine case, this follows by Lemma 2.2.0.2. □

Chapter 5

MF via groups acting on categories: Categorified Cartier/Fourier duality

5.1 Introduction

Consider the category of abelian algebraic groups $\mathbf{A}$ which are extensions of tori and discrete groups. Cartier duality is a contravariant involution of this category, taking a group to its character group

$$\mathbf{A} \leftrightarrow X^*(\mathbf{A}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbf{A}, \mathbb{G}_m)$$

It preserves finite group schemes, and interchanges the affine and discrete groups.

In this context, it is straightforward to see that:

- ($n = 0$) Algebraic functions on $X^*(A)$ = algebraic distributions on $\mathbf{A}$.
- ($n = 1$) Quasi-coherent sheaves on $X^*(A)$ = Quasi-coherent sheaves on $B\mathbf{A}$ (=vector spaces acted on by $\mathbf{A}$). Note that we could have replaced sheaves and vector spaces by complexes throughout. For $A = \mathbb{G}_m$, this is the description of $\mathbb{G}_m$-representations as $\mathbb{Z}$-graded vector spaces. For $A = \mathbb{Z}$, this is the description of a $\mathbb{Z}$-representation as a vector spaces with an automorphism.
- ($n = 2$) Quasi-coherent categories over $X^*(A)$ = Quasi-coherent categories on $B^2\mathbf{A}$ (=category acted on by $B\mathbf{A}$). For $A = \mathbb{Z}$, this is the description of a category with $B\mathbb{Z} = S^1$-action as a category with a central automorphism. As before, there are higher-categorical analogs which we will discuss in more detail below.

Though we don't wish to get into higher-categorical details, the pattern in fact continues for all n:

$$\left\{ \begin{array}{r} \text{Quasi-coherent } n\text{-categorical} \\ \text{gadgets on } X^*(A) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{r} \text{Quasi-coherent } n\text{-categorical} \\ \text{gadgets on } B^n\mathbf{A} \end{array} \right\} \left(= \left\{ \begin{array}{r} n\text{-categorical gadgets} \\ \text{acted on by } B^{n-1}\mathbf{A} \end{array} \right\} \right)$$

Here, "n-categorical gadgets" could for instance be a suitable notion of "k-linear $(\infty, n-1)$-categories." For instance, one direction of the above should be:

Theorem 5.1.0.3. *Suppose Y is an $(n-1)$-connected space and regard $C_*(\Omega^n Y)$ as an E_n-algebra. Then, there is an equivalence*

$$C_*(\Omega^n Y)\text{–mod}^n \xrightarrow{\sim} (k\text{–mod}^n)^{\Omega Y}$$

commuting, up to homotopy, with the forgetful functor to $\mathcal{C}$.

Heuristic Proof. We give a sketch, assuming the existence of a suitable notion of (∞, n)-categories enriched over k-mod. In particular, for $n = 1$, this can be made precise:

$$\begin{aligned}
(k\text{–mod}^n)^{\Omega Y} &= \mathrm{Map}_{(\infty,1)\mathbf{Cat}}\left(Y, k\text{–mod}^n\right) \\
&= \mathrm{Map}_{(\infty,n)\mathbf{Cat}}\left(Y, k\text{–mod}^n\right) \\
&= \mathrm{Map}_{(\infty,n)\mathbf{Cat}}\left(B^n_{cat}\left[\Omega^n Y\right], k\text{–mod}^n\right) \\
&= \mathrm{Map}_{(\infty,n)\mathbf{Cat}(k\text{-mod})}\left(B^n_{cat}\left[C_*(\Omega^n Y)\right], k\text{–mod}^n\right) \\
&= \mathrm{Map}_{\mathrm{mod}^{n+1}\text{-}k}\left(\mathrm{mod}^n\text{-}C_*(\Omega^n Y), k\text{–mod}^n\right) \\
&= C_*(\Omega^n Y)\text{–mod}^n
\end{aligned}$$

□

In particular, setting $Y = B^2\mathbb{Z}$ and $n = 2$, we see that

$$\left\{ \begin{array}{r} \text{quasi-coherent} \\ \text{dg-categories over } \mathbb{G}_m \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{r} \text{dg-catgories acted} \\ \text{on by } S^1 = B\mathbb{Z} \end{array} \right\}.$$

Given a category with S^1-action, one has several constructions available: coinvariants, invariants, and the Tate construction. It turns out to be possible to describe (a variant of) matrix factorizations in these terms. But the usual version of matrix factorizations involves the line or additive group, $\mathbb{G}_a$, rather than multiplicative group $\mathbb{G}_m$. To fit this into this framework, we will need to pass to an *infinitesimal* variant of the above:

Suppose $\mathbf{A}$ is an extension of vector group and formal completions of vector groups. Let $X^*(\mathbf{A}) \stackrel{\text{def}}{=} InfHom(\mathbf{A}, \mathbb{G}_a)$ be the (possibly formal) group of infinitesimal (=set-theoretically constant) characters to $\mathbb{G}_a$: If $\mathbf{A} = \mathbf{V}$ is a vector group, then $X^*(\mathbf{A})$ is the formal completion of the dual $\mathbf{V}^\vee$. If $\mathbf{A} = \widehat{\mathbf{V}}$ is the completion of a vector group, then $X^*(\mathbf{A})$ is the dual $\mathbf{V}^\vee$. Then,

- ($n = 0$) Algebraic functions on $X^*(\mathbf{A})$ = algebraic distributions on $\mathbf{A}$. This is an algebraic version of the Fourier transform, identifying functions on a line with distributions on the dual line up to finiteness conditions (thus the completions).

- ($n = 1$) Quasi-coherent sheaves on $X^*(\mathbf{A})$ = quasi-coherent sheaves on $B\mathbf{A}$ (=vector spaces acted on by $\mathbf{A}$). This is a consequence of the fact that the Fourier transform makes the commutative cocommutative Hopf algebra $k[x]$ self-dual up to finiteness issues. If $\mathbf{A} = \widehat{\mathbb{G}_a}$, this is the identification of Lie-representations of k and endomorphisms. If $\mathbf{A} = \mathbb{G}_a$, this is the identification of nilpotent Lie-representation of k and endomorphisms.

- ($n = 2$) Quasi-coherent categories over $X^*(\mathbf{A})$ = quasi-coherent categories on $B^2\mathbf{A}$(=category acted on by $B\mathbf{A}$). For $A = \widehat{\mathbb{G}_a}$, this is the identification of categories acted on by $B\widehat{\mathbb{G}_a}$ (or the dg-Lie algebra $k[+1]$) and categories with a central endomorphism. We will discuss this in more detail below.

In particular, the case we will be interested in is

$$\left\{\begin{array}{r}\text{quasi-coherent}\\ \text{dg-categories over } \mathbb{A}^1\end{array}\right\} \Leftrightarrow \left\{\begin{array}{r}\text{quasi-coherent}\\ \text{dg-categories over } B^n\widehat{\mathbb{G}_a}\end{array}\right\} \left(= \left\{\begin{array}{r}\text{dg-categories acted}\\ \text{acted on by the Lie algebra } k[+1]\end{array}\right\}\right)$$

5.2 Discrete Version: MF via invertible functions and S^1-actions

In this section, we sketch Constantin Teleman's description of matrix factorizations (for a map to $\mathbb{G}_m$ instead of $\mathbb{A}^1$) as arising from S^1-actions on complexes on the total space. We also give a variant of this replacing S^1 by $B\widehat{\mathbb{G}_a}$ which actually corresponds to map to $\mathbb{A}^1$. For simplicity we will first focus on the case where M is a scheme rather than an orbifold, and later will indicate the necessary domifications. One shows thaoOne shows that Let us prove the special case of "Theorem 5.1.0.3" that we will actually want:

Theorem 5.2.0.4. *Suppose A is a grouplike E_2-space, and regard $k[A] \overset{def}{=} C_*(A)$ as an E_2-algebra. Then, there is an equivalence of ∞-categories*

$$\{k[A]\text{-linear dg-categories}\} = \mathbf{dgcat}^{\mathrm{idm}}_{k[A]} \xrightarrow{\sim} (\mathbf{dgcat}^{\mathrm{idm}})^{BA} = \{\text{dg-categories acted on by } BA\}.$$

commuting, up to homotopy, with the forgetful functor to $\mathbf{dgcat}^{\mathrm{idm}}$. The same holds true replacing $\mathbf{dgcat}^{\mathrm{idm}}$ by $\mathbf{dgcat}^{\infty}$.

Proof. To produce the functor, we will construct a BA-action on $\mathrm{Perf}\, k[A]$ as a (right) $\mathrm{Perf}\, k[A]$ module category. The functor will then be given by $\mathcal{C} \mapsto \mathrm{Perf}\, k[A] \otimes_{\mathrm{Perf}\, k[A]} \mathcal{C}$ where now BA acts via the left-most $k[A]$-mod.

Let us compute the space of BA-actions on $\mathrm{Perf}\, k[A]$ as a $k[A]$-linear category. This is the space of pointed maps from B^2A to the simplicial set of $k[A]$-linear categories or taking iterated de-loopings:

$$\begin{aligned}\mathrm{Map}_{\mathrm{E}_0\mathbf{Spaces}}(B^2A, \mathbf{dgcat}^{\mathrm{idm}}_{k[A]}) &\simeq \mathrm{Map}_{\mathrm{E}_1\mathbf{Spaces}}(BA, \mathrm{Aut}_{\mathbf{dgcat}^{\mathrm{idm}}_{k[A]}}(\mathrm{Perf}\, k[A]))\\ &\simeq \mathrm{Map}_{\mathrm{E}_2\mathbf{Spaces}}(A, \left(\Omega^\infty \mathbf{HH}^\bullet_{k[A]}(\mathrm{Perf}\, k[A])\right)^\times)\\ &\simeq \mathrm{Map}_{\mathrm{E}_2\text{-alg}}(k[A], k[A]) \ni \mathrm{id}\end{aligned}$$

Corresponding to this element is a BA action on $\mathrm{Perf}\, k[A]$, and the above functor.

Having produced a functor, we will conclude by changing tracks and appealing to a Barr-Beck argument. We claim that $\mathrm{ev}_{\mathrm{pt}} \colon \mathrm{Fun}(B^2A, \mathbf{dgcat}^{\mathrm{idm}}) = (\mathbf{dgcat}^{\mathrm{idm}})^{BA} \to \mathbf{dgcat}^{\mathrm{idm}}$ satisfies the conditions of Barr-Beck: It is conservative since B^2A is connected. It preserves all limits and colimits since they are computed pointwise. In particular, it admits a left-adjoint L–heuristically given by left Kan extension–since $\mathbf{dgcat}^{\mathrm{idm}}$ is presentable ([L6, 4.2.3.7, 6.3.4.2]). It remains to identify the monad $\mathrm{ev}_{\mathrm{pt}} \circ L$ on $\mathbf{dgcat}^{\mathrm{idm}}$ with $\mathrm{Perf}\, k[A] \otimes -$. From the functor above, we obtain a map of monads and it is enough to prove that it is an equivalence. To do so, we use the description of L as a Kan extension

$$\mathrm{ev}_{\mathrm{pt}}(L(\mathcal{C})) \simeq \mathrm{colim}_{\mathrm{Map}_{B^2A}(\mathrm{pt},\mathrm{pt})}\, \mathcal{C} \simeq \mathrm{colim}_{\Omega B^2 A}\, \mathcal{C} \simeq \mathrm{colim}_{BA}\, \mathcal{C} = \mathcal{C}_A.$$

To complete the proof, we apply the following Lemma together with the $n = 1$ case of Theorem 5.1.0.3, thus identifying $\mathcal{C}_A$ with the compact objects of $(\mathrm{Ind}\,\mathcal{C})^{A^{\mathrm{op}}} = k[A]\text{-mod}(\mathrm{Ind}\,\mathcal{C}) =$

$\operatorname{Ind}(\operatorname{Perf} k[A] \otimes \mathcal{C})$. ☐

Remark 5.2.0.5. A similar Barr-Beck + induction argument is possible to make precise the higher n cases of Theorem 5.1.0.3. However, it requires some cardinality book-keeping in the formation of $-\mathrm{mod}^n$.

Lemma 5.2.0.6. *Suppose a simplicial group G acts on a small, idempotent complete, ∞-category $\mathcal{C}$. Then:*

(i) $(\operatorname{Ind}\mathcal{C})^G \simeq \operatorname{Ind}(\mathcal{C}_{G^{op}})$, *and in particular the former is compactly generated.*

(ii) The natural functor $i\colon (\operatorname{Ind}\mathcal{C})^G \to \operatorname{Ind}(\mathcal{C})$ *admits a compact-object preserving left adjoint* i^L*, and a colimit-preserving right adjoint* i^R*. Furthermore,* i *is conservative and so induces equivalences*

$$(\operatorname{Ind}\mathcal{C})^G = (i \circ i^L)\text{-mod}\,(\operatorname{Ind}\mathcal{C}) \qquad (\operatorname{Ind}\mathcal{C})^G = (i \circ i^R)\text{-comod}\,(\operatorname{Ind}\mathcal{C})$$

(iii) The natural functor $\mathcal{C}^G \to (\operatorname{Ind}\mathcal{C})^G$ *is fully-faithful, with essential image consisting of those objects* $x \in (\operatorname{Ind}\mathcal{C})^G$ *for which* $i(x) \in \mathcal{C} \subset \operatorname{Ind}(\mathcal{C})$*. In particular,* $\mathcal{C}_{G^{op}} \subset \operatorname{im}\mathcal{C}^G$.

(iv) There is a natural equivalence

$$\mathcal{C}^G = (i \circ i^L)\text{-mod}(\mathcal{C}).$$

Proof.

(i) Note that G acts on the presentable ∞-category $\operatorname{Ind}(\mathcal{C})$ by right-adjoint maps; their left adjoints may be taken to be the inverses of the action, i.e., the action of G^op on $\mathcal{C}$. So $(\operatorname{Ind}\mathcal{C})^G$ may be computed in Pr^R, or equivalently as the colimit of the opposite diagram in Pr^L. This opposite diagram is just the action of G^{op} on $\operatorname{Ind}(\mathcal{C})$, and it is by colimit and compact object preserving maps; so the colimit in Pr^L may be computed by taking the colimit of the (small, idempotent complete) ∞-categories of compact objects and then forming Ind. Putting this together, we obtain:

$$\underbrace{(\operatorname{Ind}\mathcal{C})^G}_{\mathrm{Pr}^R} \simeq \underbrace{\operatorname{Ind}(\mathcal{C})_{G^op}}_{\mathrm{Pr}^L} \simeq \operatorname{Ind}(\mathcal{C}_{G^{\mathrm{op}}})$$

(ii) Since G acts by equivalences, the limit $(\operatorname{Ind}\mathcal{C})^G$ may be computed in either Pr^L or Pr^R, the natural functor i is both colimit and limit preserving; since the diagram of left-adjoints is consists of compact-preserving functors, i is also compact-preserving (as in the argument for (i)). Since $\operatorname{Ind}(\mathcal{C})$ is compactly-generated, this implies the existence of left- and right-adjoints with the indicated properties.

The fact that i is conservative follows from observing that $(\operatorname{Ind}\mathcal{C})^G$ is the homotopy limit over a connected diagram (BG). Now, Lurie's Barr-Beck Theorem implies the desired equivalences.

(iii) Since $\mathcal{C} \to \operatorname{Ind}(\mathcal{C})$ is fully-faithful, the same is true for any limit. Realize the G-actions, and the natural functor, by a diagram

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\operatorname{Ind}(\mathcal{C})} \\ & \searrow & \downarrow \\ & & BG \end{array}$$

where the vertical maps are Cartesian fibrations and the horizontal map preserves Cartesian simplices (and is fully-faithful since $\mathcal{C} \to \operatorname{Ind}\mathcal{C}$ is). Then, $\mathcal{C}^G$ is explicitly given by Cartesian sections of $\widetilde{\mathcal{C}} \to BG$, while $(\operatorname{Ind}\mathcal{C})^G$ is explicitly given by Cartesian sections of $\widetilde{\operatorname{Ind}\mathcal{C}} \to BG$. Since BG is connected, to check if a Cartesian section of $\widetilde{\operatorname{Ind}\mathcal{C}} \to BG$ lands in $\widetilde{\mathcal{C}}$ it suffices to check at the base-point.

(iv) Note that the monad $i \circ i^L$ on $\operatorname{Ind}\mathcal{C}$ preserves compact objects, and so gives rise to a monad on the compact objects. Then, this follows by combining (ii) and (iii). □

Corollary 5.2.0.7. *Suppose $\mathcal{C} \in \mathbf{dgcat}^{\mathrm{idm}}$, A is a discrete abelian group, and $k[A]$ its group ring as commutative algebra. Then, the following spaces are naturally equivalent:*

- $\operatorname{Map}_{\mathrm{E}_2\mathbf{Spaces}}(A, \mathbf{HH}^\bullet(C)^\times)$;
- $\operatorname{Map}_{\mathrm{E}_2\text{-alg}}(k[A], \mathbf{HH}^\bullet(C))$;
- $\{BA\text{-actions on } \mathcal{C}\}$.

Furthermore, suppose given one of these two pieces of data. Regard $\operatorname{Perf} k$ *as commutative $k[A]$-algebra via the augmentation $A \mapsto 1$, and let $X^*(A) = \operatorname{Spec} k[A]$ as commutative group scheme. Then,*

- *There are equivalences*

$$\mathcal{C}_{BA} = \mathcal{C} \otimes_{k[A]} k \qquad \mathcal{C}^{BA} = \operatorname{Fun}^{ex}_{k[A]}(\operatorname{Perf} k, \mathcal{C})$$

  of module categories over the symmetric monoidal category $(\operatorname{Perf} k)^{BA} = \operatorname{Fun}^{ex}_{k[A]}(\operatorname{Perf} k, \operatorname{Perf} k)$.

- *This symmetric monoidal category can be identified with the convolution category* $(\operatorname{DCoh}\Omega_1 X^*(A), \circ)$.

- *If $A = \mathbb{Z}$, so that $X^*(A) = \mathbb{G}_m$, then the equivalence of completed Hopf algebras* $\exp\colon \widehat{\mathcal{O}_{\mathbb{G}_m}} \simeq \widehat{\mathcal{O}_{\mathbb{G}_a}}$ *together with Prop. 3.1.1.4 provides a symmetric monoidal equivalence* $(\operatorname{DCoh}(\Omega_1\mathbb{G}_m), \circ) \simeq (\operatorname{DCoh}(\Omega_0\mathbb{G}_a), \circ) \simeq (\operatorname{Perf} k[\![\beta]\!], \otimes_{k[\![bt]\!]})$.

Proof. The first equivalence is immediate from the previous Theorem, as is the computation of coinvariants and invariants. For the identification with the convolution category, one identifies both as subcategories of $\operatorname{Fun}^L_{k[A]}(k\text{-mod}, k\text{-mod}) \simeq \operatorname{QC}(\Omega_1 X^*(A))$. Finally, it is easy to see that this convolution category only depends on the completed Hopf algebra, proving the last claim. □

5.2.1 Hypersurfaces and S^1-actions on coherent sheaves

Lemma 5.2.1.1. *Suppose M is a (discrete) k-scheme, and $Z \subset M$ a closed subset.*

(i) Suppose that M is finite-type over k. Then, there is an equivalence of ∞-groupoids

$$H^0(M, \mathcal{O}_M)^\times = \mathbb{G}_m(M) \xrightarrow{\sim} \left\{ \begin{array}{c} S^1\text{-actions on } \mathrm{DCoh}(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

(ii) Suppose that M is finite-type over k. Then, there is an equivalence of ∞-groupoids

$$H^0(\widehat{Z}, \mathcal{O}_{\widehat{Z}})^\times = \mathbb{G}_m(\widehat{Z}) \xrightarrow{\sim} \left\{ \begin{array}{c} S^1\text{-actions on } \mathrm{DCoh}_Z(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

(iii) There is an equivalence of ∞-groupoids

$$H^0(M, \mathcal{O}_M)^\times = \mathbb{G}_m(M) \xrightarrow{\sim} \left\{ \begin{array}{c} S^1\text{-actions on } \mathrm{Perf}(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

Proof. In light of Cor. 5.2.0.7 applied with $A = \mathbb{Z}$, and the fact that $k[\mathbb{Z}]$ is discrete, it is enough to show that $\tau_{\geq 0}(\mathbf{HH}^\bullet(\mathrm{Perf}(M))) = \tau_{\geq 0}(\mathbf{HH}^\bullet(\mathrm{DCoh}(M))) = H^0(M, \mathcal{O}_M)$ and that $\tau_{\geq 0}(\mathbf{HH}^\bullet(\mathrm{DCoh}_Z(M))) = H^0(\widehat{Z}, \mathcal{O}_{\widehat{Z}})$. For then in each case

$$\mathrm{Map}_{\mathrm{E}_2\text{-alg}}(k[\mathbb{Z}], \mathbf{HH}^\bullet) = \mathrm{Map}_{\mathrm{E}_2\text{-alg}}(k[\mathbb{Z}], \tau_{\geq 0}\mathbf{HH}^\bullet)$$

is just the discrete space of commutative algebra maps, which will be the units as claimed. For $\mathrm{Perf}(M)$ this is standard, for $\mathrm{DCoh}(M)$ this is Cor. A.2.5.1. The case of $\mathrm{DCoh}_Z(M)$ we argue as follows:

$$\begin{aligned}
\Omega^\infty \mathbf{HH}^\bullet\left(\mathrm{QC}^!_Z(M)\right) &= \Omega^\infty \operatorname{RHom}_{\mathrm{QC}^!_{Z^2}(M^2)}\left(\Delta_* \mathrm{R}\Gamma_Z\, \omega_M, \Delta_* \mathrm{R}\Gamma_Z\, \omega_M\right) \\
&= \Omega^\infty \operatorname{RHom}_{\mathrm{QC}^!(M^2)}\left(\Delta_* \mathrm{R}\Gamma_Z\, \omega_M, \Delta_* \omega_M\right) \\
&= \Omega^\infty \operatorname{RHom}_{\mathrm{QC}^!(M^2)}\left(\text{“}\varinjlim_n\text{”}\, \Delta_* \mathcal{RH}om(\mathcal{O}/\mathcal{I}_Z^n, \omega_M), \Delta_* \omega_M\right) \\
&= \Omega^\infty \varprojlim_n \operatorname{RHom}_{\mathrm{DCoh}(M^2)}\left(\Delta_* \mathcal{RH}om(\mathcal{O}/\mathcal{I}_Z^n, \omega_M), \Delta_* \omega_M\right) \\
&= \Omega^\infty \varprojlim_n \operatorname{RHom}_{\mathrm{DCoh}(M^2)}\left(\Delta_* \mathcal{O}_M, \Delta_* \mathcal{O}/\mathcal{I}_Z^n\right) \\
&= \pi_0 \varprojlim_n \operatorname{RHom}_{\mathrm{DCoh}(M^2)}\left(\Delta_* \mathcal{O}_M, \Delta_* \mathcal{O}/\mathcal{I}_Z^n\right) \\
&= \varprojlim_n \pi_0 \operatorname{RHom}_{\mathrm{DCoh}(M^2)}\left(\Delta_* \mathcal{O}_M, \Delta_* \mathcal{O}/\mathcal{I}_Z^n\right) \\
&= H^0\left(\widehat{Z}, \mathcal{O}_{\widehat{Z}}\right)
\end{aligned}$$

(While the full $\mathbf{HH}^\bullet(\mathrm{QC}^!_Z(M))$ may be unwieldy, it still has no positive homotopy groups so that $\Omega^\infty = \pi_0$ admits a nice description.) □

Definition 5.2.1.2. Suppose M is a discrete finite-type k-scheme, $Z \subset M$ a closed subset, and $\widehat{Z}$ the formal completion. For $f \in \mathbb{G}_m(M)$ (resp., $f \in \mathbb{G}_m(\widehat{Z})$), define $\mathrm{CircMF}(M, f)$ (resp., $\mathrm{CircMF}(\widehat{Z}, f)$) to be $\mathrm{DCoh}(M)$ (resp., $\mathrm{DCoh}_Z(M) = \mathrm{DCoh}(\widehat{Z})$) equipped with the S^1-action of Lemma 5.2.1.1.

5.2.1.3. The previous Lemma encoded the intuitive statement that an $S^1 = B\mathbb{Z}$ action on $\mathcal{C}$ is just a compatible family of automorphisms of the Hom-spaces, here given by multiplication by $f \in \mathbb{G}_m(M)$. We now move on to showing the intuitive claim that $\mathcal{C}^{S^1}$ consists of objects equipped with trivializations of this automorphism, with maps given by the the fixed points for the induced S^1-action on mapping spaces. Since a trivialization of multiplication by f is precisely a null-homotopy of $f-1$, we will show that $(\mathrm{DCoh}\, M)^{S^1} = \mathrm{DCoh}(M_1)$ where M_1 is the derived fiber of f over $1 \in \mathbb{G}_m$. This can be viewed as explaining Prop. 3.1.2.1.

Corollary 5.2.1.4. *Suppose that M is a (discrete) k-scheme, $Z \subset M$ closed. Set $\mathcal{C} = \mathrm{Perf}(M)$ (resp., if M is finite-type, $\mathcal{C} = \mathrm{DCoh}_Z(M) = \mathrm{DCoh}(\widehat{Z})$). Via Lemma 5.2.1.1, a morphism $f\colon M \to \mathbb{G}_m$ (resp., $f\colon \widehat{Z} \to \mathbb{G}_m$) gives rise to a natural S^1-action on $\mathcal{C}$. The monad $i \circ i^L$ of Lemma 5.2.0.6 identifies with $\mathcal{O}_{M_1} = \mathcal{O}_M \otimes_{\mathcal{O}_{\mathbb{G}_m}} k \in \mathbf{Alg}(\mathrm{QC}(M))$ (resp., $\mathcal{O}_{\widehat{\mathscr{Z}}_1} = \mathcal{O}_{\widehat{\mathscr{Z}}} \otimes_{\mathcal{O}_{\mathbb{G}_m}} k \in \mathbf{Alg}(\mathrm{QC}(\widehat{Z}))$) giving rise to natural equivalences*

$$\mathcal{C}^{S^1} = \mathcal{O}_{M_1}\text{-mod}(\mathcal{C}) \qquad \text{and} \qquad \mathcal{C}_{S^1} = (\mathcal{O}_{M_1}\text{-mod}(\mathrm{Ind}\,\mathcal{C}))^c$$

(resp. $\mathcal{O}_{\widehat{\mathscr{Z}}_1}$ versions). Consequently,

- *If $\mathcal{C} = \mathrm{DCoh}(M)$ (resp. $\mathrm{DCoh}_Z(M)$), this gives $\mathcal{C}^{S^1} \simeq \mathrm{DCoh}(M_1)$ (resp., $\mathrm{DCoh}(\widehat{\mathscr{Z}_1})$). In particular, $\mathrm{DCoh}(M_1)$ (resp., $\mathrm{DCoh}(\widehat{\mathscr{Z}_1})$) is naturally $C^*(BS^1, k)$-linear.*[1]

- *If $\mathcal{C} = \mathrm{Perf}(M)$, this $\mathcal{C}_{S^1} = \mathrm{Perf}(M_1)$.*

Proof. Follows form Cor. 5.2.0.7. □

Remark 5.2.1.5. The previous two results imply a very slight refinement of the statement that $\mathrm{MF}(M, f)$ depends only on a formal completion of the (critical locus intersect the) zero fiber in M: It depends only on the ∞-category of coherent complexes on the completion, together with an S^1-action encoding the function.

5.2.1.6. If $M = U/\!\!/G$ is a global quotient orbifold, then G acts on $\mathrm{Perf}(U)$ with $\mathrm{Perf}(U)^G = \mathrm{Perf}(M)$ (by faithfully flat descent for $\mathrm{Perf}(-)$). It follows from Lemma 5.2.1.1, applied to M, that a G-invariant invertible function on M gives rise to an action of S^1 on $\mathrm{Perf}(U)$ compatible with this G-action. Thus we obtain an S^1-action on $\mathrm{Perf}(U)^G = \mathrm{Perf}(M)$, and applying Lemma 5.2.1.1 on G we see $\mathrm{Perf}(M)^{S^1} = \mathrm{DCoh}(U_1)^G = \mathrm{DCoh}(U_1/\!\!/G)$.

However, even in the global quotient case there could be *other* S^1-actions not coming from a function on the quotient. The point is that $\mathbf{HH}^0(\mathrm{Perf}(M))$ involves functors and so is naturally local on M^2, rather than M. In the scheme case this went way, but in the orbifold case the inertia stack $I_M = \pi_0 LM = \pi_0(M \times_{M^2} M)$ will intervene

Lemma 5.2.1.7. *Suppose M is an orbifold. Then, there is an equivalence of ∞-groupoids*

$$\mathrm{Ext}^0_M(\mathcal{O}_{I_M}, \mathcal{O}_M)^\times \simeq \mathbf{HH}^0_k(\mathrm{Perf}(M))^\times \xrightarrow{\sim} \left\{ \begin{matrix} S^1\textit{-actions on } \mathrm{Perf}(M) \\ \textit{as } k\textit{-linear } \infty\textit{-category} \end{matrix} \right\}$$

[1] This corresponds to the $k[\![\beta]\!]$-linear structure one gets by applying the construction of Section 4.1 with base $\mathscr{Y} = \mathbb{G}_m$ in place of $\mathbb{G}_a$. As mentioned in Cor. 5.2.0.7, the formal exponential induces an equivalence $\exp\colon \Omega_0\mathbb{G}_a \widetilde{\Omega}_1\, \mathbb{G}_m$ and so a symmetric monoidal equivalence $\mathrm{DCoh}(\Omega_1\mathbb{G}_m) \simeq \mathrm{DCoh}(\Omega_0\mathbb{G}_a) \simeq \mathrm{Perf}\, k[\![\beta]\!]$.↑

e.g., if $M = U/\!\!/G$ with U a smooth scheme and G a finite group, then the RHS is the units (for a certain product) in

$$\begin{aligned}\mathrm{Ext}^0_M(\mathcal{O}_{I_M},\mathcal{O}_M) &\simeq \left(\bigoplus_{\substack{g\in G\\ \mathrm{codim}(U^g)=0}} g\#H^0(U^g,\pi_0\mathcal{O}_{U^g})\right)^G \\ &\simeq \bigoplus_{\substack{[g] \text{ conj. class in } G\\ \mathrm{codim}(M^g)=0}} H^0(U^g,\mathcal{O}_{U^g})^{Z_G([g])}\end{aligned}$$

where the (right) G-action on the direct sum is by $(g\#a)\cdot h = h^{-1}gh\#ah$. (In case M is disconnected, we must sum over components of M^g of codimension zero.)

Proof. This follows from the computation

$$\begin{aligned}\mathbf{HH}^0_k(\mathrm{Perf}(M)) &= \mathrm{Ext}^0_{\mathrm{QC}(M^2)}(\Delta_*\mathcal{O}_M,\Delta_*\mathcal{O}_M)\\ &= \mathrm{Ext}^0_{\mathrm{QC}(LM)}(\Delta^*\Delta_*\mathcal{O}_M,\mathcal{O}_M)\\ &= \mathrm{Ext}^0_{\mathrm{QC}(LM)}(\tau_{\le 0}\Delta^*\Delta_*\mathcal{O}_M,\mathcal{O}_M)\\ &= \mathrm{Ext}^0_{\mathrm{QC}(LM)}(\mathcal{O}_{I_M},\mathcal{O}_M)\end{aligned}$$

and the following computation of Hochschild cohomology of an orbifold, which we sketch: Consider the commutative (not Cartesian) diagram

$$\begin{array}{ccccc} U & \xrightarrow{q} & U/\!\!/G & \xrightarrow{\pi} & BG \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta} \\ U^2 & \xrightarrow[q]{} & U^2/G^2 & \xrightarrow[\pi]{} & BG^2 \end{array}$$

A straightforward computation shows that

$$q^*\Delta_*\mathcal{O}_{U/\!\!/G} = \bigoplus_{g\in G} g\#(\Gamma_g)_*\mathcal{O}_U \in \mathrm{QC}(U^2)$$

The fact that $\Delta_*\mathcal{O}_{U/\!\!/G}$ is an algebra (indeed, the monoidal unit) for the convolution product on $\mathrm{QC}((U/\!\!/G)^2)$ manifests itself in the usual crossed product associative algebra structure $(g\#a)(g'\#a') = gg'\#a^{g'}a'$. It is $\mathcal{O}_{U^2}$-linear by $(a_1\otimes a_2)\cdot(g\#a) = g\#a_1a(a_2)^g$. There is a right G^2-action on this giving descent data to $\mathrm{QC}(U^2/G^2)$: Locally on a G-invariant affine piece it is $(g\#a)^{(g_1,g_2)} = g_1^{-1}gg_2\#a^{g_2}$ for $g,g_1,g_2\in G$ and $a\in\mathcal{O}_U$ (regarding $\mathcal{O}_U$ as having a right G-action in the natural way). Pulling back,

$$\Delta^*q^*\Delta_*\mathcal{O}_{U/\!\!/G} = \bigoplus_{g\in G} g\#\Delta^*(\Gamma_g)_*\mathcal{O}_U \in \mathrm{QC}(U)$$

equipped with the diagonal of the above G-action as descent data to $U/\!\!/G$.

In particular, by descent

$$\begin{aligned}\mathbf{HH}^0_k(\mathrm{Perf}(U/\!\!/G)) &= \left[\mathrm{Ext}^0_{U^2}\left(q^*\Delta_*\mathcal{O}_{U/\!\!/G}, q^*\Delta_*\mathcal{O}_{U/\!\!/G}\right)\right]^{G^2} \\ &= \left[\mathrm{Ext}^0_{U^2}\left(\bigoplus_{g\in G} g\#(\Gamma_g)_*\mathcal{O}_U, \bigoplus_{g'\in G} g'\#(\Gamma_{g'})_*\mathcal{O}_U\right)\right]^{G^2}\end{aligned}$$

a form which makes the product structure evident. Any such G^2-equivariant self-map is determined by where it sends $\mathrm{id}_G\#1$, and is in fact just right-multiplication by the image of 1. Writing this image as $\sum_{g'} g'\#\phi_{g'}$ for $\phi_{g'}\in\Gamma(\mathcal{O}_U)$, we see that it must satisfy various conditions such as $\phi_{g'}(a-a^{g'})=0$ for all $g'\in G$ and $a\in\Gamma(U)$. From these one can deduce the indicated description in terms of connected components and supports of fixed sets. However, we prefer to give a more geometric description, via essentially describing all of $\mathbf{HH}^\bullet$:

$$\begin{aligned}\mathbf{HH}^\bullet_k(\mathrm{Perf}(U/\!\!/G)) &= \mathrm{RHom}_{U/\!\!/G}\left(\Delta^*\Delta_*\mathcal{O}_{U/\!\!/G}, \mathcal{O}_{U/\!\!/G}\right) \\ &= \left[\mathrm{RHom}_U\left(q^*\Delta^*\Delta_*\mathcal{O}_{U/\!\!/G}, q^*\mathcal{O}_{U/\!\!/G}\right)\right]^G \\ &= \left[\mathrm{RHom}_U\left(\bigoplus_{g\in G} g\#\Delta^*(\Gamma_g)_*\mathcal{O}_U, \mathcal{O}_U\right)\right]^G\end{aligned}$$

In the following lines, L_gU denotes the *derived* fixed points $\mathrm{Spec}_{X^2}\,\mathcal{O}_\Delta \overset{L}{\otimes}_{\mathcal{O}_{X^2}} \mathcal{O}_{\Gamma_g}$, while $U^g=\pi_0 L_gU$ denotes the ordinary closed subscheme of fixed points.

$$= \left[\bigoplus_{g\in G}\mathrm{RHom}_U\left(\mathcal{O}_{L_gU}, \mathcal{O}_U\right)\right]^G$$

Passing to π_0:

$$\begin{aligned}\mathbf{HH}^0_k(\mathrm{Perf}(U/\!\!/G)) &= \left[\bigoplus_{g\in G}\mathrm{Ext}^0_U\left(\mathcal{O}_{L_gU}, \mathcal{O}_U\right)\right]^G \\ &= \left[\bigoplus_{g\in G}\mathrm{Ext}^0_U(\mathcal{O}_{U^g}, \mathcal{O}_U)\right]^G \\ &= \left[\bigoplus_{\substack{g\in G\\ \mathrm{codim}(U^g)=0}} H^0(U^g, \mathcal{O}_{U^g})\right]^G\end{aligned}$$

where the final equality results from noting that for a connected (discrete) closed subscheme $Z\subset U$, $\mathrm{Ext}^0_U(\mathcal{O}_Z,\mathcal{O}_U)=0$ unless Z is a connected component of U in which case $\mathrm{Ext}^0_U(\mathcal{O}_Z,\mathcal{O}_U)=H^0(Z,\mathcal{O}_Z)$. □

One can also describe the invariants for these "exotic" S^1-actions, but the description is less geometric: In the case of a global quotient, it is like a "non-commutative" fiber over $1 \in \mathbb{G}_m$ for the crossed product algebra.

Lemma 5.2.1.8. *Suppose M is an orbifold and set $\mathcal{C} = \mathrm{Perf}(M)$. Via Lemma 5.2.1.7, an element $\alpha \in \mathbf{HH}^0(\mathrm{Perf}(M))^\times$ gives rise to an S^1-action on $\mathcal{C}$. The monad $i \circ i^L$ of Lemma 5.2.0.6 identifies with $\mathcal{O}_\Delta \otimes_{\mathcal{O}_{\mathbb{G}_m}} k \in \mathbf{Alg}(\mathrm{QC}(M^2), \circ)$, where $\mathcal{O}_\Delta$ is a $k[\mathbb{Z}]$-algebra by $n \mapsto \alpha^n$ and where $\mathrm{QC}(M^2)$ is equipped with its convolution product and its "star integral transforms" action on $\mathcal{C}$. So, $\mathcal{C}^{S^1} = (\mathcal{O}_\Delta \otimes_{\mathcal{O}_{\mathbb{G}_m}} k)\text{-mod}(\mathrm{Perf}(M))$.*

In case $M = U /\!\!/ G$, and $\alpha = \sum_g f_g \in (\oplus_g H^0(U^g, \mathcal{O}_{U^g}))^G$ (with $f_g \neq 0$ only on codimension zero components), this admits a "crossed product" description

$$\mathrm{Perf}(U /\!\!/ G)^{S^1} = \left[\left(\bigoplus_{g \in G} g\#(\Gamma_g)_* \mathcal{O}_U\right) \otimes_{\mathcal{O}_{\mathbb{G}_m}} k\right]\text{-mod}(\mathrm{Perf}(U)).$$

Proof. The only thing new is the following: Suppose that $M = U /\!\!/ G$. Note that $\mathrm{Perf}(M) = \mathrm{Perf}(U)^G$ combined with Lemma 5.2.0.6(iv) give $\mathrm{Perf}(M) = (q^! q_*)\text{-mod}\,\mathrm{Perf}(U)$. We may identify the monad $q^! q_*$ with the "crossed product" algebra in endofunctors: $q^* \Delta_* \mathcal{O}_{U /\!\!/ G} \in \mathrm{QC}(U^2)$ under star convolution. The $k[\mathbb{Z}]$-action on $\Delta_* \mathcal{O}_{U /\!\!/ G}$ corresponds to a G^2-equivariant $k[\mathbb{Z}]$-action on $q^* \Delta_* \mathcal{O}_{U /\!\!/ G} \simeq q^! q_*$, where $n \in \mathbb{Z}$ acts by right multiplication by $(\sum_g g\# f_g)^n$. It follows that $\mathrm{Perf}(M)^{S^1} = (q^! q_* \otimes_{k[\mathbb{Z}]} k)\text{-mod}(\mathrm{Perf}(U))$, whence the desired formula. □

5.3 Generalities on Formal Groups acting on Categories

There are some subleties with formal group actions on categories—this is what led us to lead with the case of S^1 actions instead of $B\widehat{\mathbb{G}_a}$-actions, even though we're perfectly happy only dealing with maps to $\mathbb{A}^1$. In this section, we will discuss these subleties and show that–for the most part–they do not arise in the case of $B\widehat{\mathbb{G}_a}$. We will also mention how these subleties relate to *curved* dg-categories.

5.3.1 Definitions

Definition 5.3.1.1.

- A *derived Artin k-algebra* is an $A \in \mathbf{DRng}_k$ such that $\pi_* A$ is a finite-dimensional k-vector space, and $\pi_0 A$ is a local Artin k-algebra with residue field k. Let $\mathbf{DArt}_k$ be the full-subcategory of $\mathbf{DRng}_k$ spanned by the derived Artin rings. For any $A \in \mathbf{DArt}_k$, there is a natural $A \to k$ whose fiber will be denote $\mathfrak{m}_A$.

- A *formal moduli problem (fmp)* over k is a functor in $\mathscr{X} \in \mathrm{Fun}(\mathbf{DArt}_k, \mathbf{Sp})$ such that $\mathscr{X}(\mathrm{pt}) \simeq \mathrm{pt}$ and the natural map $\mathscr{X}(B \times_{k \oplus k[\ell]} k) \to \mathrm{fib}\{\mathscr{X}(\mathrm{Spf}\, B) \to \mathscr{X}(k \oplus k[\ell])\}$ is an equivalence for all $\ell > 0$ and all $B \to k \oplus k[\ell] \in \mathbf{DArt}_k$ (c.f., [L7, Remark 6.18]). We will call an arbitrary such functor with $\mathscr{X}(\mathrm{pt})$ contractible a *pre-fmp*. The inclusion of formal moduli problems into pre-fmp admits a left-adjoint [L5, 1.1.17], the "formal moduli completion.' A *derived formal group* G is a group object in formal moduli problems, i.e., a formal moduli problem G together with a factorization of its functor of points through $\mathbf{sGp}$. If G is a derived formal group, let BG denote the universal formal moduli problem receiving a map from $A \mapsto B(G(A))$; if $G(k \oplus k[\ell])$ is connected for $\ell > 0$, then this is already a formal moduli problem. Our motivating examples are: $\widehat{\mathbb{G}_a}$, whose functor of points is $\widehat{\mathbb{G}_a}(A) = \mathfrak{m}_A$ (viewed as a simplicial abelian group via Dold-Kan); and $B\widehat{\mathbb{G}_a}$, whose functor of points is $B\widehat{\mathbb{G}_a}(A) = B\mathfrak{m}_A$ (since it satisfies the connectivity assumption above).

Remark 5.3.1.2. Though, we will not need this, we note that $\mathbf{DArt}_k$ admits various explicit simplicial models. For instance, define simplicial categories $\mathbf{DArt}' \subset C_\infty\text{-alg}^\nu$ as follows:

– If A is an augmented algebra, $\mathfrak{m}_A$ denotes its augmentation ideal. If $\mathfrak{m}_A$ is a non-unital algebra, then $A = k \oplus \mathfrak{m}_A$ is the corresponding augmented algebra.

– Recall that a non-unital C_∞-algebra structure on a graded vector space $\mathfrak{m}_A$ is, by definition, a differential d_A on $\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_A[+1])$ that makes it into a dg-coLie coalgebra. A map of non-unital C_∞-algebras is, by definition, a map of the corresponding dg-coLie coalgebras. If $\mathfrak{m}_A, \mathfrak{m}_{A'}$ are non-unital C_∞-algebras, one has

$$\begin{aligned}\mathrm{Hom}_{C_\infty\text{-alg}}(\mathfrak{m}_A, \mathfrak{m}_{A'}) &= \mathrm{Hom}_{coLie-\mathrm{coalg}}\left(\left(\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_A[+1]), d_A\right), \left(\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_{A'}[+1]), d'_A\right)\right) \\ &= \mathrm{MC}\left(\left[\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_A[+1]), d_A\right]^\vee \otimes (\mathfrak{m}_{A'}, d_{A'})\right)\end{aligned}$$

– There is a simplicial structure by

$$\mathrm{Map}_{C_\infty\text{-alg}}(\mathfrak{m}_A, \mathfrak{m}_{A'})_p = \mathrm{Hom}_{C_\infty\text{-alg}}(\mathfrak{m}_A, \mathfrak{m}_{A'} \otimes \Omega_p) = \mathrm{MC}\left(\left[\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_A[+1]), d_A\right]^\vee \otimes \mathfrak{m}_{A'} \otimes \Omega_p\right)$$

where Ω_p is the commutative dg-algebra of algebraic differential forms on the p-simplex.

– This defines a simplicial category $C_\infty\text{-alg}^\nu$ having objects non-unital C_∞-algebras and mapping spaces as above. Let $\mathbf{DArt}'$ be the subcategory consisting of those $\mathfrak{m}_A$ which are connective, finite-dimensional as graded vector space, and for which $\pi_0 A$ is local.

– Above, we cheated and implicitly used the following fact in our notation: Given an L_∞-algebra L and a C_∞-algebra $(\mathfrak{m}_A, d_A)$, one can equip the tensor product $L \otimes \mathfrak{m}_A$ with the structure of L_∞-algebra. If $\mathfrak{m}_A$ were a dg-commutative algebra, rather than C_∞, this would be obvious: $[l_1 \otimes r_1, l_2 \otimes r_2] = [l_1, l_2] \otimes r_1 r_2$ and similarly for the higher brackets. However, the natural quasi-isomorphism of dg-operads

$$C_\infty \otimes L_\infty \longrightarrow \mathrm{Comm} \otimes L_\infty = L_\infty$$

admits a homotopy inverse by cofibrancy of L_∞.

The following Lemma asserts that $\mathbf{DArt}'_k$ models $\mathbf{DArt}_k$:

Lemma 5.3.1.3. *The Chevalley-Eilenberg complex of the dg-coLie coalgebra* $(\mathrm{coFree}^{\mathrm{coLie}}(\mathfrak{m}_A[+1]), d_A)$ *is an augmented commutative dg-algebra. The formation of this determines a functor of ∞-categories*

$$C_*^{\mathrm{coLie}}(\mathbb{L}_{k|A}) \colon N(\mathbf{DArt}'_k) \longrightarrow \mathbf{CAlg}^{\mathrm{aug}}(\mathrm{Cpx}_k) \longrightarrow \mathbf{CAlg}^{\mathrm{aug}}(k\text{-mod})$$

This functor is fully faithful with essential image $\mathbf{DArt}_k$.

5.3.2 What we're up against: One category level down

5.3.2.1. Notation for this Section: L is a dg-Lie algebra, $\widehat{BL}$ the corresponding formal moduli problem, $G_L = \Omega\widehat{BL}$ the coresponding formal group, and BG_L the *pre*-formal moduli problem $BG_L(A) = B(G_L(A))$.

There is an evident map $i\colon BG_L \to \widehat{BL}$ which is the inclusion of a connected component pointwise, and which realizes $\widehat{BL}$ as the formal moduli problem completion of BG_L.

5.3.2.2. To start with, let us think about what it means for L to act on a fixed complex $M_0 \in k$-mod in terms of formal moduli problems. Put differently, we wish to understand the underlying $(\infty, 0)$-category (throwing out non-invertible) morphisms of $(k\text{-mod})^L$. There is amiguity for two reasons: The first is the difference between BG_L and $\widehat{BL}$. The second is the following:

Suppose $M_0 \in k$-mod is a complex, and consider the pre-formal moduli problem of its deformations

$$(k\text{-mod})_{\widehat{M_0}}^{pre}(A) = \left\{ \widetilde{M} \in \mathrm{QC}(A), i^*\widetilde{M} \simeq M_0 \in k\text{-mod} \right\}$$

It turns out that this is not a formal moduli problem, but that it is only a matter of connected components – that is, it injects into its formal moduli completion. One can given an explicit description of this completion via $\mathrm{QC}^!$ [L5, 5.2.16]:

$$(k\text{-mod})_{\widehat{M_0}}(A) = \left\{ \widetilde{M} \in \mathrm{QC}^!(A), i^!\widetilde{M} \simeq M_0 \in k\text{-mod} \right\}$$

5.3.2.3. Thus, we have the following reasonable candidates:

- Take the $(\infty, 0)$-category having as objects pairs $(M_0 \in k\text{-mod}, \phi\colon BG_L \to (k\text{-mod})_{\widehat{M_0}}^{pre})$ where ϕ is a map of pre-fmp, and with morphisms given by equivalences of k-modules and homotopies of the induced maps of pre-fmp. Call this $\mathrm{Pair}_{pre,pre}$. Unravelling the definitions, we see that this is equivalent to the following data: M_0 and a compatible family of group morphisms $G_L(A) \to \mathrm{Aut}_{A\text{-mod}}(M_0 \otimes_k A)$.
- Take the $(\infty, 0)$-category having as objects pairs $(M_0 \in k\text{-mod}, \phi\colon \widehat{BL} \to (k\text{-mod})_{\widehat{M_0}}^{pre})$ where ϕ is a map of pre-fmp, and with morphisms given by equivalences of k-modules and homotopies of the induced maps of pre-fmp. Call this $\mathrm{Pair}_{fmp,fmp}$.
- There are also evident variants $\mathrm{Pair}_{fmp,pre}$ and $\mathrm{Pair}_{pre,fmp}$ according to which of BG_L and $(k\text{-mod})_{\widehat{M_0}}^{pre}$ we choose to complete to a formal moduli problem.

These are related as explained by the following Lemma.

Lemma 5.3.2.4. *There is a commutative diagram of spaces*

$$\begin{array}{ccccc} \mathrm{Pair}_{pre,pre} & \longleftarrow & \mathrm{Pair}_{fmp,pre} & & \\ \sim\downarrow & & \downarrow & & \\ \mathrm{Pair}_{pre,fmp} & \xleftarrow{\sim} & \mathrm{Pair}_{fmp,fmp} & \xrightarrow{\sim} & U(L)\text{-mod}^{\sim} \end{array}$$

If $i\colon BG_L \to \widehat{BL}$ is an equivalence (i.e., iff $\widehat{BL}(A)$ is connected for all A), then these are all equivalences. This holds, for instance, if L is connective.

Proof. Note that the equivalence in the bottom-left is formal nonsense (the definition of a formal moduli completion), and the equivalence in the bottom-right can be deduced form the theory of formal moduli problems. That the left vertical arrow is an equivalence follows because $BG_L(A)$ is connected for all A, and the map $(k\text{-mod})_{\widehat{M_0}}^{pre} \to (k\text{-mod})_{\widehat{M_0}}$ is an isomorphism on $\tau_{\geq 1}$.

If L is connective, then $\widehat{BL}(A)$ is connected for all A. Indeed, in this case $\mathfrak{m}_A \otimes L$ is again connected and the zero simplices $\mathrm{MC}(\mathfrak{m}_A \otimes L)_0$ vanish. □

5.3.2.5. Suppose now that we want to define an $(\infty, 1)$-category $(k\text{-mod})^L$ of chain complexes with L action, up to homotopy, and to relate this to the theory of formal moduli problems. Here are a few sensible options:

– Take the dg-category of dg-Lie modules over (a cofibrant dg-model of) L, and invert quasi-isomorphisms. This will be equivalent to $U(L)$-mod, where $U(L)$ is a universal enveloping E_1-algebra for (that model of) L.

– Take $\mathrm{QC}(BG_L)$: That is, for each $A \in \mathbf{DArt}$ and points $\eta \in B(G_L(A))$ we must provide an A-module M_η. For a map $(A, \eta) \to (A', \eta')$ of such, we must provide the compatibility data of $M_\eta \otimes_A A' \simeq M_{\eta'}$, and so on for higher diagrams of morphisms in **DArt**.

– Take $\mathrm{QC}^!(BG_L)$: That is, for each $A \in \mathbf{DArt}$ and points $\eta \in B(G_L(A))$ we must provide $M_\eta \in \mathrm{QC}^!(A)$. For a map $(A, \eta) \to (A', \eta')$ of such, we must provide the compatibility data of $M_\eta \overset{!}{\otimes}_A A' = \mathrm{RHom}_A(A', M_\eta) \simeq M_{\eta'}$, and so on for higher diagrams of morphisms in **DArt**.

– Take $\mathrm{QC}(\widehat{BL})$ or $\mathrm{QC}^!(\widehat{BL})$: Analogous to the above.

The following Lemma summarizes how they are related. Note that there are many other functors not indicated (e.g., adjoints of the ones drawn) to further confuse things. What's worse, one must be very careful with explicit chain models: One can obtain the natural functors $U(L)\text{-mod} \to \mathrm{QC}^!(\widehat{BL})$ and $U(L)\text{-mod} \to \mathrm{QC}(\widehat{BL})$ by taking two different localizations of *the same* strict functor of dg-categories. The relation of these is explained by the following Lemma.

Lemma 5.3.2.6. *There is a commutative diagram of* $(\infty, 1)$*-categories*

$$\begin{array}{ccccc} U(L)\text{-mod} & \xleftarrow{\sim} & \mathrm{QC}^!(\widehat{BL}) & \xrightarrow[\sim]{} & \mathrm{QC}^!(BG_L) \\ & & {\scriptstyle \otimes\omega} \big\uparrow & & {\scriptstyle \otimes\omega} \big\uparrow \\ & & \mathrm{QC}(\widehat{BL}) & \longrightarrow & \mathrm{QC}(BG_L) \end{array}$$

If $i \colon BG_L \to \widehat{BL}$ *is an equivalence (i.e., iff* $\widehat{BL}(A)$ *is connected for all* A*), then these are all equivalences. This holds, for instance, if* L *is connective.*

Proof. The top-left equivalence is [L5, 2.4.2, 3.5.1]. The fully faithfullness of the vertical arrows follows from the fully faithfulness of $\mathrm{QC}(A) \to \mathrm{QC}^!(A)$ for any $A \in \mathbf{DArt}$. The top-right equivalence follows by an argument analogous to that for Theorem 4.1.2.7.

It remains to argue that the right vertical arrow is essentially surjective: Suppose that $M \in \mathrm{QC}^!(BG_L)$. It suffices to show that for all $A \in \mathbf{DArt}$ and $\eta \in B(G_L(A))$

that $M_\eta \in \mathrm{QC}^!(A)$ lies in the essential image of $\mathrm{QC}(A)$. But the space $B(G_L(A))$ is connected so that the various M_η are all equivalent to M_{pt} where pt refers to the composite morphism $\mathrm{Spec}\, A \xrightarrow{p} \mathrm{pt} \to BG_L$. But, this composite diagram provides an equivalence $M_{\mathrm{pt}} = p^! M_0 = \omega_A \otimes p^* M_0$, proving that M_{pt} is in the essential image of $\mathrm{QC}(A)$.[2] $\square$

5.3.3 What we're actually up against

5.3.3.1. Notation for this Section: L is a dg-Lie algebra, $\widehat{BL}$ the corresponding formal moduli problem, $G_L^n = \Omega^n \widehat{BL}$ the coresponding formal E_n-group, and $B^n G_L^n$ the *pre*-formal moduli problem $B^n G_L^n(A) = B^n(G_L^n(A))$.

There is an evident map $i_n \colon B^n G_L^n \to \widehat{BL}$ which is the n-connected cover of the base-point component, and which realizes $\widehat{BL}$ as the formal moduli problem completion of $B^n G_L^n$.

We collect some basic facts analogous to those in the previous subsection:

Lemma 5.3.3.2. *For each $n \geq 1$,*

(i) There are equivalences $\mathrm{QC}(B^n G_L^n) \xrightarrow{\sim} \mathrm{QC}^!(B^n G_L^n) \xleftarrow{\sim} \mathrm{QC}^!(\widehat{BL})$ on $\mathrm{QC}^!$.

(ii) Suppose that L is $(n-2)$-connected (i.e., $\pi_i L = 0$ for $i < n-1$). Then, i_n is an equivalence.

Proof. (i) We must check that $\pi_\ell \widehat{BL}(A) = 0$ for all $A \in \mathbf{DArt}$ and $0 \leq \ell < n$. To do this, it suffices to note that (with a suitable model) $\mathrm{MC}(\mathfrak{m}_A \otimes L)_\ell = \mathrm{MC}(\mathfrak{m}_A \otimes \Omega_\ell \otimes L) = \mathrm{pt}$ for all A and $\ell \leq n$. Indeed, take a model for L with $L_i = 0$ for $i < n-1$, for $\mathfrak{m}_A$ with $(\mathfrak{m}_A)_i = 0$ for $i < 0$, and note that $(\Omega_\ell)_i = 0$ for $i < -\ell$. Thus, $(\mathfrak{m}_A \otimes \Omega_\ell \otimes L)_{-1} = 0$ for $0 \leq \ell < n$, and so is MC.

(ii) Same proofs as in the previous section. $\square$

Remark 5.3.3.3. Define the pre-fmp

$$(\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}^{pre}(A) = \left\{ \widetilde{\mathcal{C}} \in \mathbf{dgcat}^{\mathrm{idm}}_{/A}, \widetilde{\mathcal{C}} \otimes_A k \simeq \mathcal{C}_0 \right\}$$

and let

$$j \colon (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}^{pre} \longrightarrow (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}$$

be its formal moduli problem completion. This completion exists by general nonsense, but one can describe it (sort of) explicitly:

Proposition 5.3.3.4. *(i) The map*

$$(\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}^{pre}(A) \longrightarrow (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}(A)$$

is an equivalence on $\tau_{\geq 2}$: That is, only π_0 and π_1 need to be corrected.

(ii) The maps

$$B^2 \,\mathrm{Aut}_{\mathrm{Aut}(\mathcal{C}_0)}(\mathrm{id}_{\mathcal{C}_0})_{\widehat{\mathrm{id}}}^{pre} \longrightarrow B\,\mathrm{Aut}_{\mathbf{dgcat}^{\mathrm{idm}}}(\mathcal{C}_0)_{\widehat{\mathrm{id}_{\mathcal{C}_0}}}^{pre} \longrightarrow (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}^{pre}$$

[2] Note that this is just just a reformation of one of the arguments of Lemma 5.3.2.4. Indeed, it is not hard to see that the underlying $(\infty, 0)$-categories of the four things in a square here are the four Pair categories.↑

induces equivalences on $\tau_{\geq 2}$ and thus on formal moduli completions. The pre-fmp $\mathrm{Aut}_{\mathrm{Aut}(\mathcal{C}_0)}(\mathrm{id}_{\mathcal{C}_0})^{fmp}$ *is already a formal moduli problem. The underlying complex of the tangent Lie algebra of each is* $\mathbf{HH}^\bullet(\mathcal{C}_0)[+1]$.

(iii) There is a natural equivalence

$$(\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}(A) \simeq \mathrm{MC}_\bullet(\mathfrak{m}_A \otimes \mathbf{HH}^\bullet(\mathcal{C}_0)[+1]) =: \{\textit{curved } A\textit{-linear deformations of } \mathcal{C}_0\}$$

for the Lie algebra structure on $\mathbf{HH}^\bullet(\mathcal{C}_0)[+1]$ *gotten from the* E_2*-algebra structure on* $\mathbf{HH}^\bullet(\mathcal{C}_0)$*. (Note that in the previous displayed equation, the last equals sign is giving a* definition *of the space of curved deformations.)*

Proof. – It suffices to show that the map is an equivalence after taking Ω^2 pointwise. By [L5, 5.1.9] it is enough to show that $\Omega^2(\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}{}^{pre}$ is a formal moduli problem. Note that

$$\begin{aligned}\Omega^2(\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}(A) &= \mathrm{Aut}(\mathrm{id}_{\mathcal{C}_0})_{\widehat{\mathrm{id}}}(A)\\ &= \mathrm{fib}\{\mathbf{HH}^\bullet_{/A}(\mathcal{C}_0 \otimes_k A)^\times \to \mathbf{HH}^\bullet_{/k}(\mathcal{C}_0)\}\\ &\xleftarrow{\sim} \mathrm{fib}\{(\mathbf{HH}^\bullet(\mathcal{C}_0) \otimes_k A)^\times \to (\mathbf{HH}^\bullet(\mathcal{C}_0))^\times\}\\ &\xleftarrow{\sim} 1 + \Omega^\infty(\mathbf{HH}^\bullet(\mathcal{C}_0) \otimes \mathfrak{m}_A)\end{aligned}$$

where the second-to-last equivalence holds since A is perfect over k so that $A \otimes_k -$ commutes with all limits, and the last equivalence is a manifestation of the exponential. Finally, note that for any complex V the assigment

$$A \mapsto \Omega^\infty(\mathfrak{m}_A \otimes V)$$

is a formal moduli problem: One can check that directly that the conditions hold, or note that this is the fmp associated to the abelian Lie algebra $V[-1]$ (i.e., $\mathrm{MC}(\mathfrak{m}_A \otimes V[-1])_\bullet \simeq \Omega^\infty(\mathfrak{m}_A \otimes V)$).

– Clear from the proof of (i).

– For some Lie algebra structure, this is clear from the above. To demonstrate that it is the right one requires us to produce a morphism, which we will not do here but see [L5, 5.3.16, 5.3.18]. Alternatively, one should be able to produce a very explicit morphism in terms of actually constructing deformations and correspondences of dg-categories from the subspace of $\mathrm{MC}_\bullet$ corresponding to non-curved deformations. □

5.3.3.5. Next, we consider candidates for the $(\infty, 0)$-category underlying $(\mathbf{dgcat})^L$. That is, given $\mathcal{C}_0 \in \mathbf{dgcat}^{\mathrm{idm}}$ we'd like to understand what it means to give an L-action on $\mathcal{C}_0$. Rather than listing all the "reasonable" definitions as before, we will list merely the ones which are equivalent to what we want. We will call any of these a G_L action on $\mathcal{C}_0$.

– Define the $(\infty, 0)$-category of pairs $(\mathcal{C}_0, \phi\colon L \to \mathbf{HH}^\bullet(\mathcal{C})[+1])$ where ϕ is a map of dg-Lie algebras, etc.

– Define the $(\infty, 0)$-category of pairs $(\mathcal{C}_0, \phi\colon \widehat{BL} \to (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}$, where ϕ is a map of pre-fmp, etc.

– Define the $(\infty,0)$-category of pairs $(\mathcal{C}_0, \phi\colon B^nG_L^n \to (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}$, where ϕ is a map of pre-fmp, etc. for any $n \geq 1$.

– Define the $(\infty,0)$-category of pairs $(\mathcal{C}_0, \phi\colon B^nG_L^n \to (\mathbf{dgcat}^{\mathrm{idm}})_{\widehat{\mathcal{C}_0}}^{pre}$, where ϕ is a map of pre-fmp, etc. for any $n \geq 2$. For $n=2$, this may be thought of as the compatible data of maps

$$B^2(\Omega^2\widehat{BL}(A)) \longrightarrow \{\ A\text{-linear deformation of } \mathcal{C}_0\}$$

– Define the $(\infty,0)$-category of pairs $(\mathcal{C}_0, \phi\colon G_L^2 \to \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})_{\widehat{\mathrm{id}_{\mathcal{C}}}})$, where ϕ is a map of E_2-objects in pre-fmp, etc. This may be thought of as the compatible data of E_2-space maps

$$\Omega^2\widehat{BL}(A) \longrightarrow \mathbf{HH}^\bullet_{/A}(\mathcal{C}\otimes_k A)^\times \quad \text{or equivalently} \quad C_*(\Omega^2\widehat{BL}(A)) \longrightarrow \mathbf{HH}^\bullet(\mathcal{C})\otimes_k A$$

for all $A \in \mathbf{DArt}$.

Lemma 5.3.3.6. *The above are all in fact equivalent spaces.*

Proof. The first two are related by the equivalence of fmp and dg-Lie algebras. The 2nd and 3rd are equivalent formally from $B^nG^n_{\mathscr{L}} \to \widehat{BL}$ being a formal moduli completion, and the target being an fmp. The 3rd and 4th are equivalent since $B^nG^n_{\mathscr{L}}$ for $n \geq 2$ is 1-connected, and the map on the right is an equivalence on $\tau_{\geq 2}$. The 4th and 5th are equivalent pointwise by loop space theory in spaces. □

5.3.3.7. Now we wish to define candidates for the $(\infty,1)$-category $(\mathbf{dgcat}^{\mathrm{idm}})^L$ (with the obvious modifications for $(\mathbf{dgcat}^\infty)^L$). If there were an existing $(\infty,2)$-category of curved A_∞-categories over reasonable rings, that would be a good candidate. Lacking that, we'll make do with:

– Let $U_{E_2}(L)$ be the E_2-enveloping algebra of L. Then, one can consider $\mathbf{dgcat}^{\mathrm{idm}}_{U_{E_2}(L)}$.

– Let $\mathbf{dgcat}^{\mathrm{idm}}_{B^2G_L^2}$ be the $(\infty,1)$-category of quasi-coherent categories over $B^2G_L^2$: That is, for every $A \in \mathbf{DArt}$ and $\eta \in B^2(\Omega^2\widehat{BL})$ we must provide $\mathcal{C}_\eta \in \mathbf{dgcat}^{\mathrm{idm}}_A$, etc.

Theorem 5.3.3.8. *There is an equivalence* $\mathbf{dgcat}^{\mathrm{idm}}_{B^2G_L^2} \xrightarrow{\sim} \mathbf{dgcat}^{\mathrm{idm}}_{U_{E_2}(L)}$ *commuting, up to homotopy, with the forgetful functor to* $\mathbf{dgcat}^{\mathrm{idm}}$.

Proof. There is a natural equivalence

$$\mathbf{dgcat}_{B^2G_L^2} = \mathbf{dgcat}_{B^2\Omega G_L} \xrightarrow{\sim} \mathrm{Tot}\left\{\mathbf{dgcat}_k \rightleftarrows \mathbf{dgcat}_{B\Omega G_L} \rightleftarrows \mathbf{dgcat}_{(B\Omega G_L)^2} \rightleftarrows \cdots\right\}$$

and some Barr-Beck trickery identifies

$$\mathbf{dgcat}_{B^2\Omega G_L} \simeq (\mathrm{QC}(B\Omega G_L), \circ)\text{-mod}$$

where $\mathrm{QC}(B\Omega G_L)$ inherits a convolution monoidal structure $\circ$ from the remaining product on $B\Omega G_L = B\Omega^2\widehat{BL}$. By the results of the previous subsection, there is a natural equivalence $\mathrm{QC}(B\Omega G_L) = U(\Omega L)$-mod where ΩL is the loop space of L in Lie algebras.

Finally, recall that Koszul duality furnishes a natural equivalence of E_1-, co-E_∞-, bialgebras $C_*(\Omega L) = \mathrm{coBar}(C_*(L)) \simeq U(L)$. Taking cobar once more, this produces an equivalence of E_2-algebras $U(\Omega L) \simeq \mathrm{coBar}^2(C_*(L)) \simeq U_{E_2}(L)$. □

Definition 5.3.3.9. We will henceforth refer to either of the above two equivalent categories as the $(\infty, 1)$-category of "dg-categories acted on by the Lie algebra L" or "dg-categories acted on by the formal group G_L," and will denote them interchangeably $(\mathbf{dgcat}^{\mathrm{idm}})^L$ or $(\mathbf{dgcat}^{\mathrm{idm}})^{G_L}$.

Remark 5.3.3.10. Though we don't prove it here, the above can be lifted to an equivalence of $(\infty, 2)$-categories. This gives, perhaps after some unwinding, a funny putative definition of the $(\infty, 2)$-category of curved A_∞-categories over $A \in \mathbf{DArt}$: As the $(\infty, 2)$-category $\mathbf{dgcat}_{\mathrm{KD}^{\mathscr{E}_2}(A)}$ of dg-categories linear over the E_2 Koszul dual of A. With $\mathrm{QC}^!$ we were lucky, and we could define it for many more commutative rings than just those in $\mathbf{DArt}$. Unfortunately, it is not clear if the same is true for this higher-categorical version.

At this point, the reader could ask why–other than perverse aesthetics–we bothered with formal moduli problems. The point was to give a clean construction, without having to produce formulas in dg-models and check well-definedness, of things like the following:

Proposition 5.3.3.11. *Suppose L is a Lie algebra and G_L the corresponding formal group. An L-action on a category $\mathcal{C}_0$ induces an L-action on its Hochschild invariants. That is, one has a functor of $(\infty, 1)$-categories*

$$(\mathbf{HH}_\bullet, \mathbf{HH}^\bullet)\colon ((\mathbf{dgcat}^{\mathrm{idm}})^\sim)^L \longrightarrow \mathrm{E}_2^{calc}\text{-alg}(L\text{-mod})$$

Proof. For each $A \in \mathbf{CAlg}(k\text{-mod})$, one has a functor

$$(\mathbf{HH}_\bullet^{/A}, \mathbf{HH}^\bullet_{/A})\colon (\mathbf{dgcat}^{\mathrm{idm}}_{/A})^\sim \to \mathrm{E}_2^{calc}(A\text{-mod})$$

where $(\mathbf{dgcat}^{\mathrm{idm}}_{/A})^\sim$ denotes the the theory of A-linear dg-categories with non-invertible morphisms discarded, and where E_2^{calc} is 2-colored operad governing pairs of an E_2-algebra and an E_2-module over it with a circle action compatible with the circle action on the E_2-operad. (If one is concerned by issues of actually getting functoriality, see the discussion in Section 7.3.) Furthermore, this construction is functorial in A–that is, for a map $A \to A'$ there is a natural transformation $\mathbf{HH}^\bullet_{/A}(-) \otimes_A A' \to \mathbf{HH}^\bullet_{/A'}(- \otimes_A A')$. If $A \to A'$ is such that $- \otimes_A A'$ commutes with all limits (such as a finite map of Noetherian derived rings), then this natural transformation is an equivalence.[3]

Composition with this functor produces a functor

$$((\mathbf{dgcat}^{\mathrm{idm}})^\sim)^{G_L} \simeq (\mathbf{dgcat}_{B^2\Omega G_L})^\sim \overset{(\mathbf{HH}_\bullet, \mathbf{HH}^\bullet)}{\longrightarrow} \mathrm{QC}(B^2\Omega G_L) \simeq (k\text{-mod})^{G_L}.$$

Note that this preserves quasi-coherence[4] because any map $A \to A' \in \mathbf{DArt}$ is finite, so that the functor $- \otimes_A A'$ preserves limits. □

Remark 5.3.3.12. Having constructed this without formulas, we can now write down some partial formulas. This attempt may to help explain why we did not just try to define the functors above by formulas from the start.

[3]More precisely, there are two coCartesian fibrations over $\mathbf{CAlg}$ and a map between them. First, one considers the $(\infty, 1)$ category of pairs $(A, \mathcal{C})$ with $A \in \mathbf{CAlg}(k\text{-mod})$ and $\mathcal{C} \in \mathbf{dgcat}_A$ and morphisms $(A, \mathcal{C}) \to (A', \mathcal{C}')$ given by pairs of a $\mathbf{CAlg}$-map $A \to A'$ and an A-linear *equivalence* $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$; it is a coCartesian fibration via forgetting $\mathcal{C}$. Then, one considers the $(\infty, 1)$-category of pairs (A, M) where $A \in \mathbf{CAlg}(k\text{-mod})$ and $M \in \mathrm{E}_2^{calc}\text{-alg}(M)$ and maps are maps of pairs in the evident sense (with no restriction on anything being an equivalence); it is a coCartesian fibration via forgetting M.↑

[4]i.e., preserves coCartesian sections of the above fibrations↑

Let us try to describe the $\mathbf{HH}^\bullet$ part of the above in terms of curved A_∞-categories. For the remainder of this remark, nothing is up to homotopy! Fix a (actual, strict) A_∞-category $\mathcal{C}$ and consider the chain complex

$$\mathrm{coBar}(\mathcal{C}) = \prod_{n\geq 0, c_0,\dots,c_n \in \mathrm{ob}\,\mathcal{C}} \mathrm{RHom}_{\mathcal{C}}(c_0,c_1)[+1]\otimes\cdots\otimes \mathrm{RHom}_{\mathcal{C}}(c_{n-1},c_n)[+1]$$

equipped with the coBar differential $d_{\mathcal{C}}$ encoding the differentials and composition laws on $\mathcal{C}$. It is a dg-coalgebra with the evident coproduct. Then, $\mathrm{coDer}(\mathrm{coBar}(\mathcal{C}))$ is a dg-Lie algebra modelling $\mathbf{HH}^\bullet(\mathcal{C})[+1]$.

Let $A \in \mathbf{DArt}'$, so that in particular $-\otimes_k A$ commutes (on the nose) with products. An element $\eta \in \mathrm{MC}(\mathfrak{m}_A \otimes \mathrm{coDer}(\mathrm{coBar}(\mathcal{C})))$ gives rise to a (potentially curved) A-linear A_∞-category structure on $\mathcal{C}\otimes_k A$ by equipping $\mathrm{coBar}(\mathcal{C})\otimes_k A = \mathrm{coBar}_{/A}(\mathcal{C}\otimes_k A)$ with the differential $d_{\mathcal{C}}\otimes 1 + \eta$. Consequently, $\mathrm{coBar}(\mathcal{C})\otimes_k A$ equipped with the differential $d_{\mathcal{C}}\otimes 1+\eta$ will be a B_∞-algebra (see §7.5.1) in Cpx_A=the dg-category of dg-A-modules. In fact, it is in the full subcategory Cpx'_A of dg-modules whose underlying graded module over the underlying graded algebra of A is free; it can be deduced from work of Positselski that this models $\mathrm{QC}^!(A)$ (with tensor product of dg-A-modules corresponding to the shriek tensor product)!

Thus, we have defined a map of *sets*

$$\mathrm{MC}(\mathfrak{m}_A\otimes \mathrm{coBar}(\mathcal{C}))_0 \longrightarrow B_\infty\text{-alg}(\mathrm{Cpx}'_A)$$

To extend to higher simplices one defines an explicit Kan complex for $\mathrm{E}_2\text{-alg}(\mathrm{QC}^!(A))^\sim$ in terms of B_∞-algebras in Cpx'_A with coefficients in Ω_p. Having done that, one realizes that this Kan complex is none other than $\mathrm{MC}(\mathfrak{m}_A \otimes \mathrm{Der}^{\mathrm{B}_\infty}(\mathrm{coDer}(\mathrm{coBar}(\mathcal{C}))[-1]))$ and that the map one wrote down was determined by a map of dg-Lie algebras (on explicit models of) $\mathbf{HH}^\bullet(\mathcal{C})[+1] \to \mathrm{Der}_{\mathrm{E}_2}(\mathbf{HH}^\bullet(\mathcal{C}))$. We will come back to this in chapter 7, including describing this map of dg-Lie algebras in §7.5.1.

5.4 Infinitesimal Version: MF via functions and $B\widehat{\mathbb{G}_a}$-actions

Lemma 5.4.0.13. *Suppose V is a (discrete) vector space, and consider the abelian Lie algebra $V[n-1]$. Then, there is a natural identification of the E_n-enveloping algebra of V, $U_{E_n}(V)$, with the (discrete) E_n-algebra $\mathrm{Sym}_k V$.*

Proof. The evident inclusion of abelian Lie algebra $V[n-1] \hookrightarrow (\mathrm{Sym}_k V)[n-1]$ induces a map of E_n-algebras $U_{E_n}(V[n-1]) \to \mathrm{Sym}_k V$. It suffices to show that this map induces an equivalence of P_n-algebras on homotopy groups. Since V, with vanishing bracket, is a filtered algebra over the filtered algebra $\mathrm{Lie}[n-1]$, one has that $U_{E_n}(V[n-1])$ is a filtered E_n-algebra with $\mathrm{gr} U_{E_n}(V[n-1]) = U_{P_n}(\mathrm{gr} V[n-1])$. But now, there is a straightforward identification $U_{P_n}(L) = \mathrm{Sym}\, L[1-n]$ with the induced bracket.

Alternatively, we could have argued as follows: $U_{E_n}(L)$ may be computed (at least up to issues of completion) by iterated Koszul duality. That is, one forms the cocommutative coalgebra $C_*(L)$, restricts it to an E_n-coalgebra, and then forms its E_n Koszul dual algebra by iterated cobar constructions. In this case $C_*(V[n-1]) = \mathrm{coFree}^{\mathrm{coComm}}(k[+n]) \simeq H_*(B^n\mathbb{Z}, k)$ with no differential, and $\mathrm{coBar}^i(C_*(V[n-1])) \simeq H_*(B^{n-i}\mathbb{Z},k)$ for $i<n$, and finally $\mathrm{coBar}^n(C_*(k[+1])) \simeq k[x]$. ∎

Theorem 5.4.0.14. *Suppose V is a (discrete) vector space. Let $L = V[+1]$ as an abelian dg-Lie algebra, and let $B\widehat{V}(A) = B(V \otimes \mathfrak{m}_A)$ be the corresponding (derived) formal group. Let $V^\vee = \operatorname{Spec} \operatorname{Sym}_k V$ as abelian group scheme. Then, there are equivalence of the following ∞-categories*

- $\mathbf{dgcat}^{\mathrm{idm}}_{/\operatorname{Sym}_k V} = \{\mathcal{O}_{V^\vee}$*-linear dg-categories*$\}$;
- $(\mathbf{dgcat}^{\mathrm{idm}})^L = \{$*dg-categories acted on by* $V[+1]\}$ *in the sense above;*
- $\mathbf{dgcat}_{B^2\widehat{V}}$ *in the sense above: That is, for every A and $\eta \in B^2(\mathfrak{m}_A)$ we have $\mathcal{C}_\eta \in \mathbf{dgcat}^{\mathrm{idm}}_{/A}$; this is equivalent to giving a compatible family of $B(\mathfrak{m}_A)$ actions on $\mathcal{C} \otimes_k A \in \mathbf{dgcat}^{\mathrm{idm}}_A$.*

commuting, up to homotopy, with the forgetful functor to $\mathbf{dgcat}^{\mathrm{idm}}$. The same holds true replacing $\mathbf{dgcat}^{\mathrm{idm}}$ by $\mathbf{dgcat}^{\infty}$.

Proof. The equivalence of the first two follows from Theorem 5.3.3.8 and Lemma 5.4.0.13.

Since L is abelian and connected, one readily checks that $G_L = B\widehat{V}$ and $G_L^2 = \widehat{V}$–so that the last term is in fact the same as what we called $(\mathbf{dgcat}^{\mathrm{idm}})^{G_L}$ earlier. Note that the maps $B^2 G_L^2 \to BG_L \to \widehat{BL}$ are equivalences since L is 0-connected (Lemma 5.3.3.2 for $n = 2$), so that the subleties we were worried about do not arise! □

And, the infinitesimal analog of Cor. 5.2.0.7:

Corollary 5.4.0.15. *Suppose $\mathcal{C} \in \mathbf{dgcat}^{\mathrm{idm}}$, V is a (discrete) vector group, and maintain the notation of the previous Theorem. Then, the following space are naturally equivalent*

- $\{$*Curved* $C^*(V[+1]) = \widehat{\operatorname{Sym}_k \mathscr{V}^\vee}$*-linear deformations of* $\mathcal{C}\}$.
- $\operatorname{Map}_{\text{Lie-alg}}(V[+1], \mathbf{HH}^\bullet(\mathcal{C})[+1])$;
- $\operatorname{Map}_{\mathrm{E}_2\text{-alg}}(\operatorname{Sym}_k V, \mathbf{HH}^\bullet(\mathcal{C}))$;
- $\{V[+1]$*-actions on* $\mathcal{C}\}$;
- $\{B\widehat{V}$*-actions on* $\mathcal{C}\}$;

Furthermore, suppose given one of these two pieces of data. Regard $\operatorname{Perf} k$ *as commutative* $\operatorname{Sym}_k V$*-algebra via the augmentation $V \mapsto 0$, and let $V^\vee = \operatorname{Spec} \operatorname{Sym}_k V$ as commutative group scheme. Then,*

- *There are equivalences*

$$\mathcal{C}_{B\widehat{V}} = \mathcal{C} \otimes_{\operatorname{Sym}_k V} k \qquad \mathcal{C}^{B\widehat{V}} = \operatorname{Fun}^{ex}_{\operatorname{Sym}_k V}(\operatorname{Perf} k, \mathcal{C})$$

  of module categories over the symmetric monoidal category $(\operatorname{Perf} k)^{B\widehat{V}} = \operatorname{Fun}^{ex}_{\operatorname{Sym}_k V}(\operatorname{Perf} k, \operatorname{Perf} k)$.
- *This symmetric monoidal category can be identified with the convolution category* $(\operatorname{DCoh} \Omega_0 V^\vee, \circ)$.
- *If $A = k$, so that $V^\vee = \mathbb{G}_a$, then Prop. 3.1.1.4 provides a symmetric monoidal equivalence* $(\operatorname{DCoh}(\Omega_0 \mathbb{G}_a), \circ) \simeq (\operatorname{Perf} k[\![\beta]\!], \otimes_{k[\![bt]\!]})$.

Proof. All but one of the equivalences in the first part follow from the previous Theorem. For that one equivalene, we note that

$$\mathrm{Map}_{\text{Lie-alg}}(V[+1], \mathbf{HH}^\bullet(\mathcal{C})[+1]) = \mathrm{MC}_\bullet(\mathfrak{m}_{C^*(V[+1])}\widehat{\otimes}\mathbf{HH}^\bullet(\mathcal{C})[+1])$$

which was our definition of the space of curved deformation of $\mathcal{C}$. The computatoin of invariants and coinvariants follow from the identification $(\mathbf{dgcat}^{\mathrm{idm}})^{B\widehat{V}} \simeq \mathbf{dgcat}^{\mathrm{idm}}_{\mathrm{Sym}_k V}$. The symmetric monoidal identification follows by identifying both as subcategories of $\mathrm{Fun}^L_{\mathrm{Sym}_k V}(k\text{-mod}, k\text{-mod}) = \mathrm{QC}(\Omega_0 V^\vee)$. ☐

5.4.1 Hypersurfaces and $B\widehat{\mathbb{G}_a}$-actions on coherent sheaves

5.4.1.1. The use of S^1-actions gives rise to natural comparison maps $\mathbf{HH}^{k[\![\beta]\!]}_\bullet((\mathrm{Perf}\, M)^{S^1}) \to \mathbf{HH}^k_\bullet(\mathrm{Perf}\, M)^{S^1}$, etc. However, it imposes the constraint that we work with an invertible function $f\colon M \to \mathbb{G}_m$ instead of the usual superpotential $f\colon M \to \mathbb{A}^1$. If we are willing to complete near the zero fiber, it is always possible to replace f by e^f. However, completing is inconvenient in cases where we wish to retain nice global properties of M (e.g., smoothness of Perf M) and incompatible with the graded context. Thus, it is desireable to repalce replace the (constant) simplicial group S^1 by the formal group stack $B\widehat{\mathbb{G}_a}$.

We have the following analog of Lemma 5.2.1.1:

Lemma 5.4.1.2. *Suppose M is a (discrete) k-scheme, and $Z \subset M$ a closed subset.*

(i) Suppose that M is finite-type over k. Then, there is an equivalence of ∞-groupoids

$$H^0(M, \mathcal{O}_M) = \mathbb{G}_a(M) \xrightarrow{\sim} \left\{ \begin{array}{c} B\widehat{\mathbb{G}_a}\text{-actions on } \mathrm{DCoh}(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

(ii) Suppose that M is finite-type over k. Then, there is an equivalence of ∞-groupoids

$$H^0(\widehat{Z}, \mathcal{O}_{\widehat{Z}}) = \mathbb{G}_a(\widehat{Z}) \xrightarrow{\sim} \left\{ \begin{array}{c} B\widehat{\mathbb{G}_a}\text{-actions on } \mathrm{DCoh}_Z(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

(iii) There is an equivalence of ∞-groupoids

$$H^0(M, \mathcal{O}_M) = \mathbb{G}_a(M) \xrightarrow{\sim} \left\{ \begin{array}{c} B\widehat{\mathbb{G}_a}\text{-actions on } \mathrm{Perf}(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{array} \right\}$$

And:

Lemma 5.4.1.3. *Suppose that M is a smooth k-scheme. Via Lemma 5.4.1.2, a morphism $f\colon M \to \mathbb{A}^1$ gives rise to a $B\widehat{\mathbb{G}_a}$-action on $\mathcal{C} = \mathrm{Perf}(M) = \mathrm{DCoh}(M)$. Under the identification of (i), there is a natural $k[\![\beta]\!]$-linear equivalence*

$$\mathcal{C}_{B\widehat{\mathbb{G}_a}} = \mathrm{Perf}\, \mathcal{O}_{M_0} \otimes_{\mathcal{O}_M} \mathrm{Perf}(M) = \mathrm{Perf}(M_0) \quad \text{and} \quad \mathcal{C}^{B\widehat{\mathbb{G}_a}} = \mathcal{O}_{M_0}\text{-mod}(\mathrm{DCoh}(M)) = \mathrm{DCoh}(M_0)$$

where the $k[\![\beta]\!]$-linear structure on the left is as in Cor. 5.4.0.15 and on the right is as in §3.1.1.

And the orbifold variants, which are exactly analogous to what we have done above.

Lemma 5.4.1.4. *Suppose M is a an orbifold. There is an equivalence of (discrete) ∞-groupoids*

$$\mathbf{HH}^0_k(\mathrm{Perf}(M)) \simeq \mathrm{Ext}^0_M(\mathcal{O}_{I_M}, \mathcal{O}_M) \xrightarrow{\sim} \left\{ \begin{matrix} B\widehat{\mathbb{G}_a}\text{-actions on } \mathrm{Perf}(M) \\ \text{as } k\text{-linear } \infty\text{-category} \end{matrix} \right\}$$

e.g., if $M = U/\!\!/G$ with U a smooth scheme and G a finite group, then the RHS is

$$\begin{aligned}\mathrm{Ext}^0_M(\mathcal{O}_{I_M}, \mathcal{O}_M) &\simeq \left(\oplus_{\substack{g \in G \\ \mathrm{codim}(U^g)=0}} g\#H^0(U^g, \pi_0\mathcal{O}_{U^g}) \right)^G \\ &\simeq \bigoplus_{\substack{[g] \text{ conj. class in } G \\ \mathrm{codim}(M^g)=0}} H^0(U^g, \mathcal{O}_{U^g})^{Z_G([g])}\end{aligned}$$

where the (right) G-action on the direct sum is by $(g\#a) \cdot h = h^{-1}gh\#ah$. (In case M is disconnected, we must sum over components of M^g of codimension zero.)

Lemma 5.4.1.5. *Suppose M is an orbifold and set $\mathcal{C} = \mathrm{Perf}(M)$. Via Lemma 5.4.1.4, an element $\alpha \in \mathbf{HH}^0(\mathrm{Perf}(M))$ gives rise to a $B\widehat{\mathbb{G}_a}$-action on $\mathcal{C}$. The monad $i \circ i^L$ of Lemma 5.2.0.6 identifies with $\mathcal{O}_\Delta \otimes_{k[x]} k \in \mathbf{Alg}(\mathrm{QC}(M^2), \circ)$, where $\mathcal{O}_\Delta$ is a $k[x]$-algebra via $x \mapsto \alpha$ and where $\mathrm{QC}(M^2)$ is equipped with its convolution product and its "star integral transforms" action on $\mathcal{C}$. So, $\mathcal{C}^{B\widehat{\mathbb{G}_a}} = (\mathcal{O}_\Delta \otimes_{k[x]} k)\text{-mod}(\mathrm{Perf}(M))$.*

In case $M = U/\!\!/G$, and $\alpha = \sum_g f_g \in (\oplus_g H^0(U^g, \mathcal{O}_{U^g}))^G$ (with $f_g \neq 0$ only on codimension zero components), this admits a "crossed product" description

$$\mathrm{Perf}(U/\!\!/G)^{B\widehat{\mathbb{G}_a}} = \left[\left(\bigoplus_{g \in G} g\#(\Gamma_g)_*\mathcal{O}_U \right) \otimes_{k[x]} k \right]\text{-mod}(\mathrm{Perf}(U))$$

5.5 Comparison of three viewpoints

Lemma 5.5.0.6. *The inclusion $\mathbb{Z} \to \mathbb{G}_a$ induces a map of pre-fmp $\widehat{B\mathbb{Z}} \to \widehat{B\mathbb{G}_a}$ which realizes the target as the fmp-completion of the source. Furthermore, there is a natural equivalence $B\widehat{\mathbb{G}_a} = \widehat{B\mathbb{G}_a}$.*

5.5.1 Back and forth

We begin with the following variant of Lemma 5.2.1.1, which is motivated by the idea that $\mathrm{PreMF}(\mathbb{A}^1, x)$ over $k[\![\beta]\!]$ is "like" $\widehat{\mathbb{G}_a}$ over $\widehat{\mathbb{G}_a}$.

Lemma 5.5.1.1. *There is an equivalence of ∞-groupoids*

$$\left\{ \begin{matrix} S^1\text{-actions on } k\text{-mod} \\ \text{as } k[\![\beta]\!]\text{-linear category} \end{matrix} \right\} \longleftrightarrow k[\![x]\!]^\times$$

Proof. This is a variant of Lemma 5.2.1.1, using Theorem 3.2.2.3: An S^1-action on k-mod as $\mathrm{QC}^!(\mathbb{B})$-linear category, is the same as the data of a loop map

$$S^1 \xrightarrow{\otimes} \mathrm{Aut}_{\mathrm{QC}^!(\mathbb{B})}(\mathrm{QC}^!(k))$$

Since $S^1 = B\mathbb{Z}$, this is the same as giving a double loop map

$$\mathbb{Z} \xrightarrow{\otimes^2} \mathrm{Aut}_{\mathrm{Fun}^L_{\mathrm{QC}^!(\mathbb{B})}(\mathrm{QC}^!(k))}\left(\mathrm{id}_{\mathrm{QC}^!(k)}\right) \subset \mathrm{End}_{\mathrm{Fun}^L_{\mathrm{QC}^!(\mathbb{B})}(\mathrm{QC}^!(k))}\left(\mathrm{id}_{\mathrm{QC}^!(k)}\right)$$

Identify $\mathrm{QC}^!(k) = \mathrm{PreMF}^\infty(\mathbb{A}^1, -x)$ as $\mathrm{QC}^!(\mathbb{B})$-linear category. By Theorem 3.2.2.3 (and Theorem 4.1.2.8) there is an equivalence of ∞-categories

$$\mathrm{Fun}^L_{\mathrm{QC}^!(\mathbb{B})}(\mathrm{QC}^!(k), \mathrm{QC}^!(k)) = \mathrm{PreMF}^\infty_{0\times 0}(\mathbb{A}^2, -x+y) = \mathrm{QC}^!_{0\times 0}(\{x=y\}) = \mathrm{QC}^!_0\,\mathbb{A}^1 = \mathrm{QC}^!\,\widehat{0}$$

under which the identify functor corresponds to

$$\mathrm{id}_{\mathrm{QC}^!(k)} \mapsto \overline{\Delta}_*\left(\mathrm{R}\Gamma_0\,\omega_{\mathbb{A}^1}\right) \mapsto \mathrm{R}\Gamma_0\,\omega_{\mathbb{A}^1} \mapsto \omega_{\widehat{0}}$$

so that

$$\begin{aligned}\mathrm{End}_{\mathrm{Fun}^L}(\mathrm{id}_{\mathrm{QC}^!(k)}) &= \mathrm{End}_{\mathrm{QC}^!\,\widehat{0}}\left(\omega_{\widehat{0}}\right) = \Omega^\infty\,\mathrm{RHom}_{\mathrm{QC}^!\,\widehat{0}}\left(\omega_{\widehat{0}}, \omega_{\widehat{0}}\right)\\ &= \Omega^\infty\,\mathrm{RHom}_{\mathrm{QC}\,\widehat{0}}\left(\mathcal{O}_{\widehat{0}}, \mathcal{O}_{\widehat{0}}\right) = \Omega^\infty k[\![x]\!] = k[\![x]\!].\end{aligned}$$

In particular, we see that (as a 2-fold loop space) $\mathrm{Aut}(\mathrm{id}_{\mathrm{QC}^!(k)})$ identifies with the (discrete) 2-fold loop space $k[\![x]\!]^\times$. Since both $\mathbb{Z}$ and $k[\![x]\!]^\times$ are discrete, 2-fold loop maps are just the same as ordinary (abelian) group homomorphisms:

$$\mathrm{Map}_{\otimes^2}\left(\mathbb{Z}, \mathrm{Aut}_{\mathrm{id}_{\mathrm{QC}^!(k)}}\right) = \mathrm{Map}_{\otimes^2}\left(\mathbb{Z}, k[\![x]\!]^\times\right) = \mathrm{Map}_{\mathrm{AbGp}}(\mathbb{Z}, k[\![x]\!]^\times) = k[\![x]\!]^\times. \qquad \square$$

Definition 5.5.1.2. For $\varphi \in k[\![x]\!]^\times$, let $k\text{-mod}_\varphi$ and $\mathrm{Perf}\,k_\varphi$ (or just k_φ for short) denote k-mod and $\mathrm{Perf}\,k$ equipped with the S^1-action of the previous Lemma. Although we don't introduce notation for it, it should be regarded as a mixture of PreMF and CircMF with the two functions $-x$ (to $\mathbb{G}_a$) and φ (to $\mathbb{G}_m$):

$$k_\varphi = Pre/CircMF(\widehat{\mathbb{G}_a}, -x, \varphi) \qquad \widehat{\mathbb{G}_a} \xrightarrow{-x\times\varphi} \widehat{\mathbb{G}_a} \times \widehat{\mathbb{G}_m}$$

The $k[\![\beta]\!]$-action is from taking the fiber over 0 in the first variable, the S^1-action is from the second.

This allows us to incorporate S^1 actions into Cor. 3.1.2.4:

Proposition 5.5.1.3. *Suppose (M, f) is a formal LG pair, and $\varphi \in k[\![x]\!]^\times$. Set $M_0 = M \times_{\mathbb{A}^1} 0$, and $\widehat{M_0}$ the formal completion of M along M_0. Then, there is an S^1-equivariant equivalence*

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k_\varphi = \mathrm{CircMF}(\widehat{M_0}, \varphi(f))$$

Proof. At the level of underlying dg-categories,

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k_\varphi = \mathrm{PreMF}_{M_0\times 0}(M \times \mathbb{A}^1, f \boxplus -x) = \mathrm{DCoh}_{M_0}(\Gamma_f(M)) = \mathrm{DCoh}(\widehat{M_0})$$

Consider the diagram

$$\widehat{M_0} \xrightarrow{\Gamma_f} \widehat{M_0} \times \widehat{\mathbb{G}_a} \xrightarrow{p_2} \widehat{\mathbb{G}_a} \xrightarrow{\varphi} \widehat{\mathbb{G}_m}$$

The S^1-action comes from the second projection $\varphi\colon \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}$, i.e., $\varphi(f)$. ☐

Finally, we sketch a few of the compatibilities between the various constructions we have seen:

Proposition 5.5.1.4. *Suppose M is a smooth formal k-scheme and $f \in \mathbb{G}_m(M)$. Then, there is a $k[\![\beta]\!]$-linear equivalence*

$$\mathrm{CircMF}(M, f)^{S^1} \simeq \mathrm{PreMF}(\widehat{M_1}, \log(f))$$

where $\log(f) = \log(1 + (f-1)) = \sum (-1)^m (f-1)^m/m$.

Proof. Consider the diagram

$$\widehat{M_1} \xrightarrow{f} \widehat{\mathbb{G}_m} \xrightarrow[\sim]{\log} \widehat{\mathbb{G}_a}$$

and note that the second map is an equivalence of abelian formal groups. Now combine Section 4.1 and Cor. 5.2.1.4 (with its footnote). ☐

Proposition 5.5.1.5. *Suppose (M, f) is an LG pair, and $\varphi \in k[x] \cap k[\![x]\!]^\times$ with $\varphi(0) = 1$. Shrinking M if necessary, suppose that $\varphi(f) \in \mathbb{G}_m(M)$ so that both* $\mathrm{CircMF}(M, \varphi(f))$ *and* $\mathrm{PreMF}(M, f)$ *make sense. Then,*

(i) There is a $k[\![\beta]\!]$-linear equivalence

$$\mathrm{CircMF}(M, \varphi(f))^{S^1} = \mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} (k_\varphi)^{S^1}$$

(ii) If $\varphi'(x) \neq 0$, then $(k_\varphi)^{S^1}$ is an invertible $k[\![\beta]\!]$-module category (in fact, equivalent to $\mathrm{Perf}\, k[\![\beta]\!]$*).*

Proof.

(i) The inclusion $\mathrm{CircMF}(\widehat{M_0}, \varphi(f)) \to \mathrm{CircMF}(M, \varphi(f))$ induces an equivalence on S^1-fixed points. By Prop. 5.5.1.3 it remains to check that the natural map

$$\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} (k_\varphi)^{S^1} \to (\mathrm{PreMF}(M, f) \otimes_{k[\![\beta]\!]} k_\varphi)^{S^1}$$

is an equivalence. This will follow from (ii) upon noting that the underlying k-linear category on both sides identifies with $\mathrm{DCoh}(M_0)$ by Cor. 5.2.1.4, and the k-linear functor with the identity functor.

(ii) Consider the diagram

$$\widehat{\mathbb{G}_a} \xrightarrow{\Delta} \widehat{\mathbb{G}_a} \times \widehat{\mathbb{G}_a} \xrightarrow{\mathrm{id} \times \varphi} \widehat{\mathbb{G}_a} \times \widehat{\mathbb{G}_m}$$

Using Prop. 5.5.1.4, there is a $k[\![\beta]\!] \otimes k[\![\beta]\!]$-linear identification of $(k_\varphi)^{S^1}$ with DCoh on the fiber over 0×1. By hypothesis, $\mathrm{id} \times \varphi$ is an isomorphism of formal groups, so this identifies with $\mathrm{PreMF}(\widehat{\mathbb{G}_a}, x, x)$. ☐

Remark 5.5.1.6. If (M, f) is an LG pair, then $\mathrm{Perf}(M)$ with the $B\widehat{\mathbb{G}_a}$-action corresponding to f (or $\mathrm{CircMF}(M, f)$ in the case of $f \in \mathbb{G}_m(M)$) remembers information about all the fibers of f. An easier version of the construction in this section tells us how: The space of $B\widehat{\mathbb{G}_a}$-actions (resp., S^1-actions) on $\mathrm{Perf}\, k$ identifies with $\Gamma(\mathrm{pt}, \mathcal{O}_{\mathrm{pt}}) = k$ (resp., $k^\times$). For $t \in k$ (resp., $\lambda \in k^\times$) let k_t (resp., k_λ) denote this. Then, we can twist the formation of invariants by k_t (resp., k_λ)

$$\mathrm{Fun}_{B\widehat{\mathbb{G}_a}}(k_t, \mathrm{Perf}(M)) = (k_{-t} \otimes \mathrm{Perf}(M))^{B\widehat{\mathbb{G}_a}} = \mathrm{PreMF}(M, f - t)$$

One could in principle hope for a refined version of Theorem 6.1.2.5, not factoring through taking invariants on the category level, which retains information about finer global invariants (e.g., non-commutative Hodge structures). However, the trick then lies in forgetting *some* information so thatthe construction is not trivial!

Chapter 6

Applications: Hochschild invariants (unstructured), smoothness, etc.

6.1 Smooth, proper, CY, and HH

6.1.1 Smoothness (and properness) of MF

Using Theorem 3.2.2.3, we are able to obtain the show that MF is smooth, and that it is proper when the critical locus is proper:

Theorem 6.1.1.1 (Smoothness and Properness)**.** *Suppose (M, f) is an LG pair, $Z \subset f^{-1}(0)$ closed. Then,*

(i) Suppose Z_{red} is proper. Then, $\mathrm{PreMF}_Z(M, f)$ is proper over $k[\![\beta]\!]$ and $\mathrm{MF}_Z(M, f)$ is proper over $k(\!(\beta)\!)$.

(ii) Suppose Z contains each connected component of $\mathrm{crit}(f)$ which it intersects. Then, $\mathrm{MF}_Z(M, f)$ is smooth over $k(\!(\beta)\!)$.

(iii) Suppose $\mathrm{crit}(f) \cap f^{-1}(0)$ is proper. Then, $\mathrm{MF}(M, f)$ is smooth and proper over $k(\!(\beta)\!)$.

Proof.

(i) It suffices to show that ev, restricted to compact objects, factors through $\mathrm{Perf}\, k[\![\beta]\!]$. Unraveling, it suffices to verify that

$$\mathrm{ev}(\mathscr{F} \otimes \mathscr{G}) = \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}(M,f)}(\mathscr{F}, \mathscr{G}) \in k[\![\beta]\!]\text{-mod}$$

is perfect for all $\mathscr{F}, \mathscr{G} \in \mathrm{PreMF}_Z(M, f)$. By Prop. 3.1.2.1

$$\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}(M,f)}(\mathscr{F}, \mathscr{G}) = \mathrm{RHom}_{\mathrm{DCoh}(M)}(i_*\mathscr{F}, i_*\mathscr{G})^{S^1}$$

where $i\colon M_0 \to M$. Regarding $\mathrm{Hom}_{\mathrm{DCoh}(M)}(i_*\mathscr{F}, i_*\mathscr{G})$ with its S^1-action as a $k[B]/B^2$-module, it suffices by Prop. 3.1.1.4 to show that it is t-bounded and coherent over $k[B]/B^2$, or equivalently perfect over k.

Thus, it is enough to show that $\mathrm{RHom}_{\mathrm{DCoh}(M)}(\mathscr{F}', \mathscr{G}') \in \mathrm{Perf}\, k$ for any $\mathscr{F}', \mathscr{G}' \in \mathrm{DCoh}_Z(M)$. Let $k\colon Z_{\mathrm{red}} \to M$ be the inclusion. By Lemma 2.2.0.2 we are reduced to the case where $\mathscr{G}' = k_*\overline{\mathscr{G}}$ for some $\overline{\mathscr{G}} \in \mathrm{DCoh}(Z_{\mathrm{red}})$. Since M is regular, $\mathscr{F}'$ is perfect and hence the pullback $k^*\mathscr{F}'$ is also perfect so that $\mathcal{RH}om(k^*\mathscr{F}', \overline{\mathscr{G}}) \in \mathrm{DCoh}(Z_{\mathrm{red}})$. But now, Z_{red} is proper so that

$$\begin{aligned}\mathrm{RHom}_{\mathrm{DCoh}(M)}(\mathscr{F}', k_*\overline{\mathscr{G}}) &= \mathrm{RHom}_{\mathrm{DCoh}(Z_{\mathrm{red}})}(k^*\mathscr{F}', \overline{\mathscr{G}}) \\ &= \mathrm{R}\Gamma\left(Z_{\mathrm{red}}, \mathcal{RH}om_{\mathrm{DCoh}(Z_{\mathrm{red}})}(k^*\mathscr{F}', \overline{\mathscr{G}})\right) \in \mathrm{Perf}\, k\end{aligned}$$

as desired.

(ii) We must prove that $\mathrm{id}_{\mathrm{MF}_Z(M,f)}$ is a compact object in the functor category. An object in a $k[\![\beta]\!]$-linear ∞-category is compact iff it is compact in the category, viewed as a plain ∞-category. Using Theorem 3.2.2.3, we are reduced to showing that

$$\overline{\omega_{\Delta,Z}} = \overline{\Delta}_* \left(\underline{\mathrm{R}\Gamma}_Z \omega_M\right) \in \mathrm{MF}^{\infty}_{Z^2}(M^2, -f \oplus f) = \mathrm{Ind}\,\mathrm{DCoh}_{Z^2}((M^2)_0)\widehat{\otimes}_{k[\![\beta]\!]} k(\!(\beta)\!)$$

is compact. Since ω_M is coherent and Δ proper, $\overline{\omega_\Delta} = \overline{\Delta}_*\omega_M$ is coherent and so compact in $\mathrm{PreMF}^\infty(M^2, -f \boxplus f) = \mathrm{Ind}\,\mathrm{DCoh}((M^2)_0)$. Note that $\overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M = \underline{\mathrm{R}\Gamma}_{Z^2}\overline{\Delta}_*\omega_M$, since $Z \subset M_0$, so that it is only the $\underline{\mathrm{R}\Gamma}_{Z^2}$ that can cause problems.

Let $W = \mathrm{crit}(-f \oplus f) \cap (M^2)_0$ be the components of the critical locus of $-f \oplus f$ lying in the zero fiber. By Prop. 3.2.1.6, the natural inclusions

$$\begin{array}{ccc} \mathrm{DCoh}_{Z^2\cap W}((M^2)_0) & \hookrightarrow & \mathrm{DCoh}_{Z^2}((M^2)_0) \\ \downarrow & & \downarrow \\ \mathrm{DCoh}_W((M^2)_0) & \hookrightarrow & \mathrm{DCoh}(M^2) \end{array}$$

induce, upon applying $\mathrm{Ind}(-)\widehat{\otimes}_{k[\![\beta]\!]} k(\!(\beta)\!)$

$$\begin{array}{ccc} \underline{\mathrm{R}\Gamma}_{Z^2\cap W}(\overline{\omega_\Delta}) \in \mathrm{MF}^{\infty}_{Z^2\cap W}(M^2, -f \boxplus f) & \xrightarrow{\sim} & \mathrm{MF}^{\infty}_{Z^2}(M^2, -f \boxplus f) \ni \underline{\mathrm{R}\Gamma}_{Z^2}(\overline{\omega_\Delta}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{R}\Gamma}_W(\overline{\omega_\Delta}) \in \mathrm{MF}^{\infty}_{W}(M^2, -f \boxplus f) & \xrightarrow{\sim} & \mathrm{MF}^{\infty}(M^2, -f \boxplus f) \ni \overline{\omega_\Delta} \end{array}$$

The functors in this diagram are left adjoints, whose right adjoints are the appropriate $\underline{\mathrm{R}\Gamma}_{-}$ functors. Using the top row, we see that it suffices to show that $\underline{\mathrm{R}\Gamma}_{Z^2\cap W}(\overline{\omega_\Delta})$ is compact in $\mathrm{MF}^{\infty}_{Z^2\cap W}(M^2, -f \boxplus f)$. Using the bottom row, we see that $\underline{\mathrm{R}\Gamma}_W(\overline{\omega_\Delta})$ is compact in $\mathrm{MF}^{\infty}_W(M^2, -f \boxplus f)$. It thus suffices to show that $\underline{\mathrm{R}\Gamma}_{Z^2\cap W} \colon \mathrm{MF}^{\infty}_W(M^2, -f \boxplus f) \to \mathrm{MF}^{\infty}_{Z^2\cap W}(M^2, -f \boxplus f)$ preserves compact objects; the property of preserving compact objects is preserved under $-\widehat{\otimes}_{k[\![\beta]\!]} k(\!(\beta)\!)$, so it suffices to show that $\underline{\mathrm{R}\Gamma}_{Z^2\cap W} \colon \mathrm{Ind}\,\mathrm{DCoh}_W((M^2)_0) \to \mathrm{Ind}\,\mathrm{DCoh}_{Z^2\cap W}((M^2)_0)$ preserves compact objects. But, our assumptions on Z imply that $Z^2 \cap W$ is a union of connected components of W: so, $\mathrm{R}\Gamma_{Z^2\cap W}$ may be identified with the restriction to those connected components, and in particular preserves compact objects.

(iii) Set $Z = \mathrm{crit}(f) \cap f^{-1}(0)$. By (i) and (ii), $\mathrm{MF}_Z(M, f)$ is smooth and proper over $k[\![\beta]\!]$. By Prop. 3.2.1.6, the inclusion induces an equivalence $\mathrm{MF}_Z(M, f) \simeq \mathrm{MF}(M, f)$. □

Remark 6.1.1.2. It seems likely that the Theorem remains true if (M, f) is replaced by a *formal LG pair*: i.e., a relative DM stack $f \colon \mathscr{X} \to \widehat{\mathrm{pt}} = \mathrm{Spf}\,\widehat{\mathcal{O}_{\mathbb{A}^1}} \subset \mathbb{A}^1$ over $\widehat{\mathrm{pt}}$ with $\mathscr{X}$

formally smooth. However, the methods of this paper seem to be insufficient for this beyond the algebrizable case.

6.1.2 Hochschild-type Invariants

6.1.2.1. Suppose (M, f) is an LG pair. The $B\widehat{\mathbb{G}_a}$-action on $\mathrm{DCoh}(M)$ corresponding to f via Section 5.4 provides a $B\widehat{\mathbb{G}_a}$-action on $\mathbf{HH}_\bullet(\mathrm{DCoh}(M))$ and $\mathbf{HH}^\bullet(\mathrm{DCoh}(M))$, and by naturality maps

$$\mathbf{HH}_\bullet^{k[\![\beta]\!]}\left(\mathrm{DCoh}(M)^{B\widehat{\mathbb{G}_a}}\right) \longrightarrow \mathbf{HH}_\bullet^{k}(\mathrm{DCoh}(M))^{B\widehat{\mathbb{G}_a}}$$

and

$$\mathbf{HH}^\bullet_k(\mathrm{DCoh}(M))^{B\widehat{\mathbb{G}_a}} \longrightarrow \mathbf{HH}^\bullet_{k[\![\beta]\!]}\left(\mathrm{DCoh}(M)^{B\widehat{\mathbb{G}_a}}\right)$$

preserving all the structures naturally present on Hochschild invariants ($\mathrm{SO}(2)$-action on $\mathbf{HH}_\bullet$, E_2-algebra structure on $\mathbf{HH}^\bullet$, and $\mathbf{HH}^\bullet$-module structure on $\mathbf{HH}_\bullet$ suitably compatible with the $\mathrm{SO}(2)$-action).

6.1.2.2. The goal of this section will, roughly, be to study the degree to which these are equivalences. Our main tool will be an alternate description of these actions. Equip M^2 with the difference $-f \boxplus f$, so that $(M^2)_0 = \{(m_1, m_2) : f(m_1) = f(m_2)\}$. Let $\ell\colon (M^2)_0 \to M^2$ be the natural inclusion. The discussion of chapter 3, in particular Prop. 3.1.2.1, equips the complexes

$$\mathbf{HH}_\bullet(\mathrm{DCoh}(M)) = \mathrm{RHom}_{\mathrm{DCoh}(M^2)}\left(\ell_*\overline{\Delta}_*\mathcal{O}_M, \ell_*\overline{\Delta}_*\omega_M\right)$$
$$\mathbf{HH}^\bullet(\mathrm{DCoh}(M)) = \mathrm{RHom}_{\mathrm{DCoh}(M^2)}\left(\ell_*\overline{\Delta}_*\mathcal{O}_M, \ell_*\overline{\Delta}_*\mathcal{O}_M\right)$$

with S^1-actions. Under the identification of S^1- and $B\widehat{\mathbb{G}_a}$-actions on complexes, and the identification of Cor. A.2.5.1, these are equivalent to the functorially induced actions of 6.1.2.1.

Before moving on to our goal, let us decribe this S^1-action in more detail. In chapter 7, we will show that this this action–in a way compatible with the higher algebraic structure–is determined purely by the algebraic structure itself: It is given by the adjoint action by $f \in \mathbf{HH}^0(M)$ in the Lie algebra $\mathbf{HH}^\bullet(M)[+1]$. At the level of complexes–ignoring the higher structure–we can see this directly:

Proposition 6.1.2.3. *Suppose (M, f) is an LG pair with M a scheme. Then, the HKR identifications $\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(M)) \simeq \Omega^\bullet_M$ and $\underline{\mathbf{HH}}^\bullet(\mathrm{Perf}(M)) \simeq T^\bullet_M = \oplus \bigwedge^i T_M[-i]$ lift to S^1-equivariant identifications, where the S^1-actions on the right are given by $-df \wedge -$ and $i_{df}(-)$.*

Proof. We prove the result in the affine case $M = \operatorname{Spec} R$; the proof globalizes in the same way as HKR itself, by completing the cyclic bar complexes, etc. Recall the cyclic bar-type resolution of $\Delta_*\mathcal{O}_M = R$ as $\mathcal{O}_{M^2} = R \otimes R$-bimodule

$R \longrightarrow \left|R^{\otimes(\bullet+2)}\right|$ degeneracies given by inserting 1, face maps given by multiplying adjacent elements.

It remains to give this a structure of $\mathcal{O}_{M^2}[B_{M^2}]$-module quasi-isomorphic to $\overline{\Delta}_*\mathcal{O}_M$. We claim that this can be explicitly done by setting

$$B_{M^2}(a_1 \otimes \cdots a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i \otimes f \otimes a_{i+1} \otimes \cdots \otimes a_n$$

Indeed, a straightforward computation verifies that $B_{M^2}^2 = 0$ and $d(B_{M^2} \cdot x) = (-f \boxplus f) \cdot x + B_{M^2} \cdot dx$, where d is the internal differential on the cyclic complex. Regarding $\Delta_* T$ as an $\mathcal{O}_{M^2}[B_{M^2}]$-module via the augmentation $\mathcal{O}_{M^2}[B_{M^2}] \to \mathcal{O}_{M^2}/(-f \boxplus f)$, 3.1.3.4 tells us that the S^1-action on

$$\mathcal{RH}om(\Delta_* \mathcal{O}_M, \Delta_* T) = \mathcal{RH}om_A(\Delta^* \Delta_* \mathcal{O}_M, T) = \mathrm{Tot}\left\{ \mathcal{RH}om_A(A^{\otimes(\bullet+1)}, T) \right\}$$

is simply dual to $B = B_{M^2} \otimes_{\mathcal{O}_{M^2}} \mathcal{O}_M$ in the first variable. Finally, it suffices to observe that the usual HKR map intertwines $-df \wedge -$ and B: We compute

$$\begin{aligned} B = B_{M^2} \otimes_{\mathcal{O}_{M^2}} \mathcal{O}_M \,(a_1 \otimes \cdots \otimes a_m) &= \sum_{i=1}^{m-1} (-1)^i a_1 \otimes \cdots \otimes a_i \otimes f \otimes a_{i+1} \otimes \cdots \otimes a_m \\ &\quad + (-1)^m a_1 \otimes \cdots \otimes a_m \otimes f \end{aligned}$$

so that

$$\begin{aligned} HKR\,(B\,(a_1 \otimes \cdots \otimes a_m)) &= \frac{1}{m!} \left[\sum_{i=1}^{m} (-1)^i a_1 \wedge \cdots da_i \wedge df \wedge da_{i+1} \right] \\ &= -\frac{df}{m!} \wedge (m a_1 da_2 \wedge \cdots da_m) \\ &= -df \wedge HKR\,(a_1 \otimes \cdots \otimes a_m) \end{aligned}$$

The analogous operator on $\underline{\mathbf{HH}}^\bullet$ is dual, which is $i_{df}(-)$. □

Remark 6.1.2.4. Expanding on the proof of of Lemma 5.2.1.7, and noting that the natural map $L(U^g) \to L_g U$ is an equivalence,[1] one obtains a natural equivalence

$$L(U /\!\!/ G) = (\oplus_{g \in G} L_g U) /\!\!/ G \simeq (\oplus_{g \in G} L(U^g)) /\!\!/ G$$

So that

$$\mathbf{HH}_\bullet(U /\!\!/ G) = \mathrm{R}\Gamma\left(L(U /\!\!/ G), \mathcal{O}_{L(U /\!\!/ G)}\right) = \left[\oplus_{g \in G} \mathrm{R}\Gamma(\mathcal{O}_{L_g U})\right]^G = \left[\oplus_{g \in G} \mathrm{R}\Gamma(\mathcal{O}_{L(U^g)})\right]^G$$

$$\mathbf{HH}^\bullet(U /\!\!/ G) = \mathrm{R}\Gamma\left(L(U /\!\!/ G), \omega_{L(U /\!\!/ G)/U /\!\!/ G}\right) = \left[\oplus_{g \in G} \mathrm{RHom}_U(\mathcal{O}_{L_g U}, \mathcal{O}_U)\right]^G = \left[\oplus_{g \in G} \mathrm{RHom}_U(\mathcal{O}_{L(U^g)}, \mathcal{O}_U)\right]^G$$

Identifying $\mathcal{O}_{L(U^g)} = \underline{\mathbf{HH}}_\bullet(\mathrm{Perf}\, U^g)$ and using its HKR description, one obtains *an* HKR de-

[1] It is evidently an equivalence on π_0, both terms being identified with U^g. So it suffices to verify that we have an equivalence on cotangent complexes. Applying Luna's Slice Theorem, one sees that U^g is smooth and that its cotangent bundle is the $\langle g \rangle$-invariant piece in the $\Omega_U|_{U^g}$; in particular, the conormal bundle (say at each point) contains only non-trivial $\langle g \rangle$ representations so that g acts invertibly on the conormal bundle. One can identify the cotangent complex of $L_g U$ with the cone of the action of g on $\mathbb{L}_U$; the cotangent complex of $L(U^g)$ with the cone of the zero map on $\mathbb{L}_{U^g}$; and the map of cotangent complexes with the pullback $\mathbb{L}_U|_{U^g} \to \mathbb{L}_{U^g}$. Since both U and U^g are smooth, this restriction map is surjective and its kernel is the conormal bundle of U^g in U. It thus suffices to recall that g acts invertibly on the conormal bundle.↑

scription of these orbifold Hochschild invariants.[2] Presumably a similar explicit computation is possible, though we have not tried to carry it out.

Finally, using Theorem 3.2.2.3 we are able to complete the computation of Hochschild-type invariants:

Theorem 6.1.2.5 (Hochschild-type Invariants)**.** *Suppose (M, f) is an LG pair with, and $Z \subset f^{-1}(0)$ a closed set. Then,*

(i) There are natural $k[\![\beta]\!]$-linear equivalences

$$\mathbf{HH}_\bullet^{k[\![\beta]\!]}(\mathrm{PreMF}_Z(M,f)) = \mathbf{HH}^k_\bullet(\mathrm{DCoh}_Z(M))^{S^1}$$

and

$$\mathbf{HH}^\bullet_{k[\![\beta]\!]}(\mathrm{PreMF}_Z(M,f)) = \mathbf{HH}_k^\bullet(\mathrm{DCoh}_Z(M))^{S^1}$$

(where the circle action is given, under the HKR isomorphism, by $-df \wedge -$ and not *the usual B-operator) The descriptions as invariants are compatible with the B-operator on $\mathbf{HH}_\bullet$ (=de Rham differential), and the E_2-algebra structure on $\mathbf{HH}^\bullet$, and the $\mathbf{HH}^\bullet$-module structure on $\mathbf{HH}_\bullet$.*

(ii) There is a natural $k(\!(\beta)\!)$-linear equivalence

$$\mathbf{HH}_\bullet^{k(\!(\beta)\!)}(\mathrm{MF}_Z(M,f)) = \mathbf{HH}^k_\bullet(\mathrm{DCoh}_Z(M))^{\mathrm{Tate}}$$

(iii) Either assume 0 is the only critical value of f, or set

$$\mathrm{MF}^{\mathrm{tot}} = \bigoplus_{\lambda \in \mathrm{cval}(f)} \mathrm{MF}^\infty(X, f-\lambda).$$

Then, there are natural $k(\!(\beta)\!)$-linear equivalences

$$\mathbf{HH}_\bullet^{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}\right) = \Big(\mathbf{HH}^k_\bullet(\mathrm{DCoh}\, M)\Big)^{\mathrm{Tate}}$$

$$\mathbf{HH}^\bullet_{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}\right) = \left(\mathbf{HH}_k^\bullet(\mathrm{DCoh}\, M)\right)^{\mathrm{Tate}}$$

The description in terms of Tate-cohomology of an S^1-action on the Hochschild complex of $\mathrm{DCoh}(M)$ is compatible with: the B-operator on $\mathbf{HH}_\bullet$, the E_2-algebra structure on $\mathbf{HH}^\bullet$, the $\mathbf{HH}^\bullet$-module structure on $\mathbf{HH}_\bullet$. Given a volume form on M inducing a CY structure on $\mathrm{MF}(M, f)$ (see Theorem 6.1.3.4 below) the description is compatible with the resulting BV-algebra structure on $\mathbf{HH}^\bullet$.

(iv) Suppose furthermore that M is a scheme. Then, HKR induces equivalences

$$\mathbf{HH}_\bullet^{k[\![\beta]\!]}(\mathrm{PreMF}_Z(M,f)) \simeq \mathrm{R}\Gamma_Z\left([\Omega^\bullet_M[\![\beta]\!], \beta \cdot (-df \wedge -)]\right)$$

$$\mathbf{HH}_\bullet^{k(\!(\beta)\!)}(\mathrm{MF}_Z(M,f)) \simeq \mathrm{R}\Gamma_Z\left([\Omega^\bullet_M(\!(\beta)\!), \beta \cdot (-df \wedge -)]\right)$$

$$\mathbf{HH}_\bullet^{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}\right) \simeq \mathrm{R}\Gamma\left([\Omega^\bullet_M(\!(\beta)\!), \beta \cdot (-df \wedge -)]\right)$$

[2] Presumably there is a different HKR-type description where one stops at L_gU, so that the normal bundles of U^g appears explicitly.↑

$$\mathbf{HH}^{\bullet}_{k(\!(\beta)\!)}\left(\mathrm{MF}^{\mathrm{tot}}\right) \simeq \mathrm{R}\Gamma\left(\left[\bigwedge^{\bullet} T_M[1](\!(\beta)\!), \beta \cdot i_{df}(-)\right]\right)$$

$$\mathbf{HC}^{k[\![\beta]\!]}_{\bullet}\left(\mathrm{PreMF}_Z(M,f)\right) \simeq \mathrm{R}\Gamma_Z\left([\Omega^{\bullet}_M[\![\beta,u]\!], \beta\cdot(-df\wedge -) + u\cdot d]\right)$$

$$\mathbf{HC}^{k(\!(\beta)\!)}_{\bullet}\left(\mathrm{MF}_Z(M,f)\right) \simeq \mathrm{R}\Gamma_Z\left([\Omega^{\bullet}_M(\!(\beta)\!)[\![u]\!], \beta\cdot(-df\wedge -) + u\cdot d]\right)$$

$$\mathbf{HC}^{k(\!(\beta)\!)}_{\bullet}\left(\mathrm{MF}^{\mathrm{tot}}\right) \simeq \mathrm{R}\Gamma\left([\Omega^{\bullet}_M(\!(\beta)\!)[\![u]\!], \beta\cdot(-df\wedge -) + u\cdot d]\right)$$

Proof.

(i) Let $k\colon (M^2)_0 \to M^2$ be the inclusion, $\Delta\colon M \to M^2$ the diagonal, and $\overline{\Delta}\colon M \to (M^2)_0$ the reduced diagonal. By Theorem 3.2.2.3,

$$\mathbf{HH}^{k[\![\beta]\!]}_{\bullet}\left(\mathrm{PreMF}_Z(M,f)\right) = \mathrm{ev}(\mathrm{id}_{\mathrm{PreMF}^{\infty}_Z(M,f)}) = \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^{\infty}(M^2,-f\oplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M\right)$$

Since ω_M is coherent, the standard formula for local cohomology shows that we may write

$$\overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M = \varinjlim_{\alpha} \mathscr{K}_\alpha$$

as a uniformly t-bounded filtered colimit of compacts. Then, applying Prop. 3.1.2.1:

$$\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^{\infty}(M^2,-f\oplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M\right) = \varinjlim_{\alpha}\left[\mathrm{RHom}_{\mathrm{DCoh}(M^2)}(\Delta_*\mathcal{O}_M, k_*\mathscr{K}_\alpha)^{S^1}\right]$$

By t-boundedness of the $\mathscr{K}_\alpha$, and regularity of M^2, we see that $\left\{\mathrm{RHom}_{\mathrm{DCoh}(M^2)}(\Delta_*\mathcal{O}_M, k_*\mathscr{K}_\alpha)\right\}_\alpha$ will be uniformly t-bounded. Since taking S^1-invariants commutes with uniformly t-bounded colimits, we obtain

$$= \left[\varinjlim_{\alpha}\mathrm{RHom}_{\mathrm{DCoh}(M^2)}(\Delta_*\mathcal{O}_M, k_*\mathscr{K}_\alpha)\right]^{S^1}$$

$$= \left[\mathrm{RHom}_{\mathrm{QC}^!(M^2)}(\Delta_*\mathcal{O}_M, \Delta_*\underline{\mathrm{R}\Gamma}_Z\omega_M)\right]^{S^1}$$

which by Cor. A.2.5.1 we may identify with

$$= \left[\mathbf{HH}^{k}_{\bullet}(\mathrm{DCoh}_Z(M))\right]^{S^1}$$

Analogously,

$$\begin{aligned}
\mathbf{HH}^{\bullet}_{k[\![\beta]\!]}\left(\mathrm{PreMF}_Z(M,f)\right) &= \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^{\infty}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M, \overline{\Delta}_*\underline{\mathrm{R}\Gamma}_Z\omega_M\right) \\
&= \mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^{\infty}(M^2,-f\boxplus f)}\left(\text{“}\varinjlim_{\alpha}\text{”}\mathscr{K}_\alpha, \text{“}\varinjlim_{\alpha'}\text{”}\mathscr{K}_{\alpha'}\right) \\
&= \varprojlim_{\alpha}\varinjlim_{\alpha'}\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}^{\infty}(M^2,-f\boxplus f)}\left(\mathscr{K}_\alpha, \mathscr{K}_{\alpha'}\right) \\
&= \varprojlim_{\alpha}\varinjlim_{\alpha'}\left[\mathrm{RHom}_{\mathrm{QC}^!(M^2)}\left(\mathscr{K}_\alpha, \mathscr{K}_{\alpha'}\right)\right]^{S^1}
\end{aligned}$$

As before, the t-boundedness of $\mathscr{K}_{\alpha'}$ and the regularity of M^2 imply that we may commute $\varinjlim_{\alpha'}$ past the invariants. Finally, we commute the $\varprojlim_{\alpha}$ past the invariants, to obtain

$$= \left[\mathrm{RHom}_{\mathrm{QC}^!(M^2)}\left(\Delta_* \underline{\mathrm{R}\Gamma}_Z \omega_M, \Delta_* \underline{\mathrm{R}\Gamma}_Z \omega_M\right)\right]^{S^1}$$

which by Cor. A.2.5.1 we may identify with

$$= \left[\mathbf{HH}^\bullet_k(\mathrm{DCoh}_Z(M))\right]^{S^1}$$

Recall that the compatibility with the various structures follows from the argument of 6.1.2.1.

(ii) Follows from (i) since $\mathbf{HH}_\bullet$ is compatible with the symmetric monoidal functor $- \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$.

(iii) The computation follows in a manner analogous to (i) from Theorem 3.2.2.3(v). For the Hochschild cohomology computation, it is important that the identity functor is represented by a compact object in $\mathrm{MF}^\infty(M^2, -f \boxplus f)$: This lets us avoid the question of commuting $\varprojlim_{\alpha}$ past a Tate construction. (This was the reason that (ii) above did not include a statement about Hochschild cohomology.)

(iv) We first prove the first equality: From (i), we must identify $\mathbf{HH}^k_\bullet(\mathrm{DCoh}_Z(M))$, compute the S^1-action on it, and then conclude. By Cor. A.2.5.1, $\mathbf{HH}^k_\bullet(\mathrm{DCoh}_Z(M)) = \mathrm{R}\Gamma_Z\, \underline{\mathbf{HH}}_\bullet(\mathrm{DCoh}(M))$. Since M is regular, $\mathrm{DCoh}(M) \simeq \mathrm{Perf}(M)$ and HKR identifies this inner term (de Rham complex) and its B operator (de Rham differential). Then, Prop. 6.1.2.3 identifies the circle action with $-df \wedge -$. (That this identification can be made compatibly with the B operator can be checked explicitly, but in any case will follow from chapter 7.) Finally, the desired computation follows by noting that $\mathrm{R}\Gamma_Z$ is a right adjoint and so commutes with homotopy limits, e.g., taking S^1-invariants:

$$\begin{aligned}\left[\mathbf{HH}^k_\bullet(\mathrm{DCoh}_Z(M))\right]^{S^1} &= \left[\mathrm{R}\Gamma_Z\left([\oplus_i \Omega^\bullet_M, 0]\right)\right]^{S^1} \\ &= \left[\mathrm{R}\Gamma_Z\left([\Omega^\bullet_M, 0]^{S^1}\right)\right] \\ &= \mathrm{R}\Gamma_Z\left([\Omega^\bullet_M[\![\beta]\!], \beta(-df \wedge -)]\right)\end{aligned}$$

The second equality follows from the first, since $\otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$ is monoidal, upon noting that $R\Gamma_Z$ commutes with the filtered colimit of inverting β. The third and fourth equality follow analogously from (iii) and Prop. 6.1.2.3. ☐

Remark 6.1.2.6. The presence of support conditions, and the existence of a comparison map, has a down-to-earth description in terms of Prop. 3.1.2.1 and Lemma 6.1.2.7: Use an explicit cyclic bar construction to write (leaving the differentials implicit)

$$\mathbf{HH}_\bullet(\mathrm{DCoh}(M)) = \bigoplus_{n \geq 1} \bigoplus_{c_1, \ldots, c_n \in \mathrm{DCoh}(M)} \mathrm{RHom}_M(c_1, c_2) \otimes_k \cdots \otimes_k \mathrm{RHom}_M(c_n, c_1)[n-1]$$

Then, Lemma 2.2.0.2 and Morita-invariance of $\mathbf{HH}_\bullet$ give quasi-isomorphisms

$$\begin{aligned}\mathbf{HH}_\bullet(\mathrm{DCoh}_Z(M)) &\simeq \bigoplus_{n\geq 1}\bigoplus_{c_1,\dots,c_n\in \mathrm{DCoh}_Z(M)} \mathrm{RHom}_M(c_1,c_2)\otimes_k \cdots \otimes_k \mathrm{RHom}_M(c_n,c_1)[n-1] \\ &\simeq \bigoplus_{n\geq 1}\bigoplus_{c'_1,\dots,c'_n\in \mathrm{Coh}_Z(M_0)} \mathrm{RHom}_M(i_*(c'_1),i_*(c'_2))\otimes_k \cdots \otimes_k \mathrm{RHom}_M(i_*(c'_n),i_*(c'_1))[n-1]\end{aligned}$$

We thus obtain a natural map

$$\begin{aligned}&\mathbf{HH}_\bullet(\mathrm{DCoh}_Z(M))^{S^1} \\ &\quad\longleftarrow \bigoplus_{n\geq 1}\left(\bigoplus_{c'_1,\dots,c'_n\in \mathrm{Coh}_Z(M_0)} \mathrm{RHom}(i_*(c'_1),i_*(c'_2))\otimes_k \cdots \otimes_k \mathrm{RHom}(i_*(c'_n),i_*(c'_1))\right)^{S^1}[n-1] \\ &\quad= \bigoplus_{n\geq 1}\bigoplus_{c'_1,\dots,c'_n\in \mathrm{Coh}_Z(M_0)} \mathrm{RHom}(i_*(c'_1),i_*(c'_2))^{S^1}\otimes_{k[\![\beta]\!]} \cdots \otimes_{k[\![\beta]\!]} \mathrm{RHom}(i_*(c'_n),i_*(c'_1))^{S^1}[n-1] \\ &\quad= \mathbf{HH}_\bullet^{k[\![\beta]\!]}(\mathrm{PreMF}_Z(M,f))\end{aligned}$$

at least upon verifying that the above identifications are compatible with the differentials (i.e., that Prop. 3.1.2.1 plays well with composition). As already implicit in the above, the inner direct sum is uniformly t-bounded and so commutes with $(-)^{S^1}$. From this perspective, it is not clear why the outer direct should also commutes with $(-)^{S^1}$; this is some sort of "convergence" statement about the cyclic bar complex.

Lemma 6.1.2.7. *Suppose V, V' are t-bounded complexes with S^1-action, and $V\otimes_k V'$ their tensor product as complex with S^1-action. Then, the natural map*

$$V^{S^1}\otimes_{k[\![\beta]\!]}(V')^{S^1} \longrightarrow (V\otimes_k V')^{S^1}$$

is an equivalence.

Proof. C.f., the proof of Prop. 3.1.1.4. □

6.1.3 Calabi-Yau structures on MF

We first recall the notion of a Calabi-Yau structure on a smooth, not necessarily proper, dg-category (e.g., [L8, Def. 4.2.6 & Remark 4.2.17]):

Definition 6.1.3.1. Suppose $\mathcal{C}\in \mathbf{dgcat}_R^{\mathrm{idm}}$ is smooth. An *m-Calabi-Yau structure* on $\mathcal{C}$ is an SO(2)-invariant cotrace

$$\mathrm{cotr}\colon R\to \mathrm{ev}\circ\mathrm{coev}[-m](R) = \mathbf{HH}_\bullet(\mathcal{C})[-m]$$

satisfying the following *non-degeneracy condition*:

- Note that cotr gives rise to a 1-morphism in $\mathrm{Fun}_R^L(R\text{-mod}, R\text{-mod})\simeq R\text{-mod}$:

  $$\mathrm{cotr}(V) = \mathrm{id}_V\otimes_R \mathrm{cotr}\colon\ \mathrm{id}(V)\simeq V\otimes_R R\longrightarrow V\otimes_R \mathbf{HH}_\bullet(\mathcal{C})[-m]\simeq \mathrm{ev}\circ\mathrm{coev}[-m](V)$$

- The non-degeneracy condition is that this be the co-unit of an adjunction $(\mathrm{coev}[-m], \mathrm{ev})$, i.e., that the composite

$$\mathrm{Map}_{\mathrm{Fun}^L_R(\mathrm{Ind}\,\mathcal{C},\mathrm{Ind}\,\mathcal{C})}(\mathrm{coev}(V)[-m],\mathscr{F}) \xrightarrow{\mathrm{ev}} \mathrm{Map}_{R\text{-mod}}\left(\mathrm{ev}\circ\mathrm{coev}(V)[-m],\mathscr{F}\right) \xrightarrow{\mathrm{cotr}} \mathrm{Map}_{R\text{-mod}}\left(V,\mathrm{coev}\,\mathscr{F}\right)$$

be an equivalence for all $V \in R$-mod and $\mathscr{F} \in \mathrm{Fun}^L_R(\mathrm{Ind}\,\mathcal{C},\mathrm{Ind}\,\mathcal{C})$. Since $\mathcal{C}$ is smooth, it suffices to check that this condition is verified for $V = R[n]$, $n \in \mathbb{Z}$, and $\mathscr{F} \in \left(\mathrm{Fun}^L_R(\mathrm{Ind}\,\mathcal{C},\mathrm{Ind}\,\mathcal{C})\right)^c$ compact.

Of course the motivating example is:

Lemma 6.1.3.2. *Suppose that M is an m-dimensional Calabi-Yau variety (in the weak sense that M is Gorenstein and $\omega_M[-m]$ is trivializable), and that $\mathrm{vol}_M\colon \mathcal{O}_M \simeq \omega_M[-m]$ is a holomorphic volume form. Then, vol_M gives rise to an m-Calabi-Yau structure on $\mathrm{DCoh}(M)$ as follows:*

- *There is a $\mathrm{cotr}_{[\mathrm{vol}_M]}\colon k \to \mathbf{HH}^\bullet(\mathrm{DCoh}(M))[-m]$ determined by*

  $$[\mathrm{vol}_M] = \Delta_*\,\mathrm{vol}_M \in \mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m]) = \mathrm{Map}_{k\text{-mod}}(k, \mathbf{HH}_\bullet(\mathrm{DCoh}(M))[-m])$$

- *There is a natural $\mathrm{SO}(2)$-invariant lift of $[\mathrm{vol}_M]$, which determines $\mathrm{SO}(2)$-equivariance data for $\mathrm{cotr}_{\mathrm{vol}_M}$, and $\mathrm{cotr}_{\mathrm{vol}_M}$ is non-degenerate in the above sense.*

Furthermore, this determines a bijection between equivalence classes of the m-Calabi-Yau structures and the set of holomorphic volume forms.

Proof. By assumption, $\Delta_*\mathcal{O}_M$ and $\Delta_*\omega_M[-m]$ are both coherent sheaves, i.e., the heart of the t-structure. It follows that

$$\begin{aligned}
\mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m]) &= \Omega^\infty\,\mathrm{RHom}_{M^2}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m]) \\
&= \mathrm{Ext}^0_{M^2}(\Delta_*\mathcal{O}_M, \omega_M[-m]) \\
&= \mathrm{Ext}^0_{M}(\mathcal{O}_M, \omega_M[-m]) \\
&= \mathrm{Map}_{M}(\Delta_*\mathcal{O}_M, \omega_M[-m])
\end{aligned}$$

and that both spaces are discrete. Any homotopy $\mathrm{SO}(2)$-action on a discrete space is trivial, so that

$$\pi_0\left(\mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m])^{\mathrm{SO}(2)}\right) \simeq \mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m])$$

This shows that $[\mathrm{vol}_M]$ lifts to $\mathrm{SO}(2)$-invariants. The same argument shows that vol_M is an isomorphism iff $\Delta_*\,\mathrm{vol}_M$ is so, proving the “Furthermore.” In fact, we didn't need the discreteness argument:
Claim: Δ_* admits a natural factorization through $\mathrm{SO}(2)$-invariants

$$\Delta_*\colon\ \mathrm{RHom}_M(\mathcal{O}_M, \omega_M[-m]) \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(M^2)}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m])^{\mathrm{SO}(2)}.$$

Assuming the claim, we now complete the proof: We must show that $\mathrm{cotr}_{\mathrm{vol}_M}$ is non-degenerate, i.e., that the composite

$$\mathrm{Map}_{\mathrm{Fun}^L_k(\mathrm{QC}^!(M),\mathrm{QC}^!(M))}(\mathrm{coev}(V)[-m],\mathscr{F}) \xrightarrow{\mathrm{ev}} \mathrm{Map}_{k\text{-mod}}\left(\mathrm{ev}\circ\mathrm{coev}(V)[-m],\mathscr{F}\right) \xrightarrow{\mathrm{cotr}} \mathrm{Map}_{k\text{-mod}}\left(V,\mathrm{ev}\,\mathscr{F}\right)$$

is an equivalence for all $V \in k$-mod and $\mathscr{F} \in \mathrm{Fun}_k^L(\mathrm{QC}^!(M), \mathrm{QC}^!(M))^c \simeq \mathrm{DCoh}(M^2)$. By Theorem A.2.2.4, we know that $\mathrm{DCoh}(M)$ is smooth; so, it suffices to verify the condition for $V = k[n]$, $n \in \mathbb{Z}$, and $\mathscr{F}$ compact. Using the identification of Theorem A.2.2.4, we may identify the relevant map with (global sections of shifts of)

$$\mathcal{RH}om_{\mathrm{DCoh}(M^2)}(\Delta_* \omega_M[-m], \mathscr{F}) \longrightarrow \mathcal{RH}om_{\mathrm{DCoh}(M^2)}(\Delta_* \mathcal{O}_M, \mathscr{F})$$

given by pre-composing with the equivalence $\Delta_* \mathrm{vol}_M$.

Finally, we include two proofs of the claim: the first by general nonsense for which we do not give all the details, and the second much more concrete in case M is smooth:
First Proof of Claim:

Base-change for the diagram

$$\begin{array}{ccc} LM & \xrightarrow{p_2} & M \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \Delta} \\ M & \xrightarrow[\Delta]{} & M^2 \end{array}$$

identifies

$$\mathrm{RHom}_{\mathrm{QC}^!(M^2)}(\Delta_* \mathscr{F}, \Delta_* \mathscr{G}) = \mathrm{RHom}_{LM}\left((p_2)^* \mathscr{F}, (p_1)^! \mathscr{G}\right)$$

so that in particular

$$\mathrm{RHom}_{\mathrm{QC}^!(M^2)}(\Delta_* \mathcal{O}_M, \Delta_* \omega_M[-m]) = \mathrm{RHom}_{LM}(\mathcal{O}_{LM}, \omega_{LM}[-m])$$

Let $s : M \to LM$ be the inclusion of constant loops, which is naturally $\mathrm{SO}(2)$-equivariant. Under the above, Δ_* is identified with

$$\mathrm{RHom}_M(\mathcal{O}_M, \omega_M[-m]) \xrightarrow{s_*} \mathrm{RHom}_{LM}(s_* s^* \mathcal{O}_{LM}, s_* s^! \omega_{LM}[-m]) \xrightarrow{\mathrm{tr}_s \circ (-) \circ \mathrm{unit}_s} \mathrm{RHom}_{LM}(\mathcal{O}_{LM}, \omega_{LM}[-m])$$

The lift to $\mathrm{SO}(2)$-invariants is provided by naturality from the $\mathrm{SO}(2)$-equivariance of s.

Second Proof of Claim:
In case M is smooth we can be completely explicit: By HKR, we may identify $\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(M)) = \bigoplus_i \Omega_M^i[i]$ and the $\mathrm{SO}(2)$-action with the de Rham differential d_{DR}. Then, the lift of Δ is

$$\mathrm{R}\Gamma(\omega_M[-m]) = R\Gamma(\boxed{\Omega_M^m}) \longrightarrow \mathrm{R}\Gamma \begin{pmatrix} \boxed{\Omega_M^m} & & & \\ \Omega_M^{m-1} \xrightarrow{d} \beta\Omega_M^m & & & \\ \Omega_M^{m-2} \xrightarrow{d} \beta\Omega_M^{m-1} \xrightarrow{d} \beta^2\Omega_M^m & & & \\ \cdots \quad \cdots \quad \cdots & & & \ddots \\ \mathcal{O}_M \xrightarrow{d} \beta\Omega_M^1 \xrightarrow{d} \beta^2\Omega_M^2 \xrightarrow{d} \ddots & & & \\ \beta\mathcal{O}_M \xrightarrow{d} \beta^2\Omega_M^1 \xrightarrow{d} \ddots & & & \end{pmatrix} = (\mathbf{HH}_\bullet(\mathrm{Perf}\, M)[-m])^{S^1}$$

where the boxed entries are in degree 0. The induced map on mapping *spaces* is Ω^∞ of this, which is just the identity on $H^0(\Omega^m_M)$. $\square$

Remark 6.1.3.3. The cotrace can also be made very explicit in the Dolbeault model (over $\mathbb{C}$) for Hochschild homology: Represent vol_M by a holomorphic $(n,0)$-form, $[\mathrm{vol}_M] \in \Gamma(A^{n,0})$. Then, $[\mathrm{vol}_M]$ is visibly a cycle in

$$(\mathbf{HH}_\bullet(\mathrm{Perf}\, M)[-m])^{\mathrm{SO}(2)} = \left[(\oplus_{p,q}\Gamma(\mathcal{A}^{p,q})[p-q-m])\,[\![u]\!], \overline{\partial} + u\cdot\partial\right]$$

Indeed $\overline{\partial}$ vanishes since $[\mathrm{vol}_M]$ is holomorphic, and ∂ vanishes since it is is an $(n,0)$-form.

We now come to the main result of the section.

Theorem 6.1.3.4 (Calabi-Yau structures)**.** *Suppose (M,f) is an LG pair, $m = \dim M$, and that M is equipped with a volume form $\mathrm{vol}_M \colon \mathcal{O}_M \simeq \omega_M[-m]$. Then, vol_M determines an m-Calabi-Yau structure on $\mathrm{MF}(M,f)$.*

Proof. Replacing M by an open subset, we may suppose for simplicity that 0 is the only critical point of M. For the remainder of the proof, let $\mathrm{MF} = \mathrm{MF}(M,f)$, and $\mathrm{Fun}^L = \mathrm{Fun}^L_{k(\!(\beta)\!)}(\mathrm{MF},\mathrm{MF})$ which we will identify with $\mathrm{MF}(M^2, -f \boxplus f)$ via Theorem 3.2.1.3 (with the support condition dropped by the reasoning of Theorem 6.1.1.1). Let $k\colon (M^2)_0 \to M^2$ be the inclusion.

The m-Calabi-Yau structure on $\mathrm{DCoh}(M)$ corresponding to vol_M

$$[\mathrm{vol}_M] = \Delta_*\,\mathrm{vol}_M \in \mathbf{HH}_\bullet(\mathrm{DCoh}(M))[-m] = \mathrm{RHom}_{\mathrm{DCoh}(M^2)}(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m])$$

admits the refinement

$$\overline{[\mathrm{vol}_M]}^{S^1} = \overline{\Delta}_*[\mathrm{vol}_M] \in \mathrm{RHom}_{\mathrm{PreMF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \overline{\Delta}_*\omega_M[-m]\right) = \mathrm{RHom}_{M^2}\left(\Delta_*\mathcal{O}_M, \Delta_*\omega_M[-m]\right)^{S^1}$$

which upon inverting β gives an element

$$\overline{[\mathrm{vol}_M]}^{\mathrm{Tate}} \stackrel{\mathrm{def}}{=} \overline{\Delta}_*[\mathrm{vol}_M] \in \mathbf{HH}_\bullet^{k(\!(\beta)\!)}(\mathrm{MF})[-m] = \mathrm{RHom}_{\mathrm{MF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \overline{\Delta}_*\omega_M[-m]\right)$$

Claim 1: There is an $\mathrm{SO}(2)$-action on $\mathrm{RHom}_{\mathrm{PreMF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M, \overline{\Delta}_*\omega_M[-m]\right)$ refining the natural $\mathrm{SO}(2)$-action on $\mathbf{HH}_\bullet^{k(\!(\beta)\!)}(\mathrm{MF})$.

Claim 2: The description as a pushforward via $\overline{\Delta}$ equips $\overline{[\mathrm{vol}_M]}^{S^1}$ (and so $\overline{[\mathrm{vol}_M]}^{\mathrm{Tate}}$) with a natural lift to $\mathrm{SO}(2)$-invariants.

Assuming the claims for now we complete the proof: From the claim, it follows that $\overline{[\mathrm{vol}_M]}^{\mathrm{Tate}}$ determines an $\mathrm{SO}(2)$-invariant cotrace

$$\mathrm{cotr}_{\mathrm{vol}_M} \colon k(\!(\beta)\!)[m] \longrightarrow \mathbf{HH}_\bullet^{k(\!(\beta)\!)}(\mathrm{MF})$$

We will be done if we can prove that $\mathrm{cotr}_{\mathrm{vol}_M}$ is *non-degenerate* in the sense that it is the unit for an adjunction $(\mathrm{coev}[-m], \mathrm{ev})$, i.e., that the composite

$$\mathrm{Map}_{\mathrm{Fun}^L}(\mathrm{coev}[-m](V), \mathscr{F}) \to \mathrm{Map}_{k(\!(\beta)\!)\text{-mod}}(\mathrm{ev}\circ\mathrm{coev}[-m](V), \mathrm{ev}\,\mathscr{F}) \stackrel{\mathrm{id}_V\otimes\mathrm{cotr}^\vee_{\mathrm{vol}_M}}{\longrightarrow} \mathrm{Map}_{k(\!(\beta)\!)\text{-mod}}(V, \mathrm{ev}\,\mathscr{F})$$

is an equivalence for all $V \in k(\!(\beta)\!)$-mod and $\mathscr{F} \in \mathrm{Fun}^L$. Since MF is smooth (Theorem 6.1.1.1) it suffices to verify this condition when $V = k(\!(\beta)\!)[n]$, $n \in \mathbb{Z}$ and $\mathscr{F} \in \mathrm{Fun}^L$ is compact. Using the identifications of Theorem 3.2.2.3, we see that it suffices to check that

$$\begin{aligned}
&\mathrm{RHom}_{\mathrm{MF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\omega_M[-m],\mathscr{F}\right)\\
&\qquad\longrightarrow \mathrm{RHom}_{k(\!(\beta)\!)\text{-mod}}\left(\mathrm{RHom}_{\mathrm{Fun}^L}\left(\overline{\Delta}_*\mathcal{O}_M,\overline{\Delta}_*\omega_M[-m]\right),\mathrm{RHom}_{\mathrm{Fun}^L}\left(\overline{\Delta}_*\mathcal{O}_M,\mathscr{F}\right)\right)\\
&\qquad\longrightarrow \mathrm{RHom}_{k(\!(\beta)\!)\text{-mod}}\left(k(\!(\beta)\!),\mathrm{RHom}^{k(\!(\beta)\!)}_{\mathrm{MF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M,\mathscr{F}\right)\right)\\
&\qquad= \mathrm{RHom}^{k(\!(\beta)\!)}_{\mathrm{MF}(M^2,-f\boxplus f)}\left(\overline{\Delta}_*\mathcal{O}_M,\mathscr{F}\right)
\end{aligned}$$

is an equivalence for all $\mathscr{F}\in\mathrm{MF}(M^2,-f\boxplus f)$. The composite is just given by pre-composition with $\overline{[\mathrm{vol}_M]}^{\mathrm{Tate}}$, so it suffices to observe that $\overline{[\mathrm{vol}_M]}^{\mathrm{Tate}}$ is an equivalence: It is the image by a functor of $\overline{[\mathrm{vol}_M]}^{S^1}=\overline{\Delta}_*\,\mathrm{vol}_M$, and vol_M is an equivalence.

Proof of Claims: Let $B\widehat{\mathbb{G}_a}$ act on $\mathrm{Perf}(M)$ corresponding to f (Lemma 5.4.1.2), and on $\mathrm{Perf}(M^2)$ corresponding to $-f\boxplus f$. By functoriality we know that $B\widehat{\mathbb{G}_a}$-acts on $\mathbf{HH}^k_\bullet(\mathrm{Perf}(M))$ compatibly with the SO(2)-action, and this induces an SO(2) action on

$$\mathbf{HH}^k_\bullet(\mathrm{Perf}(M))^{B\widehat{\mathbb{G}_a}}=\mathrm{RHom}_{M^2}(\Delta_*\mathcal{O}_M,\Delta_*\omega_M)^{B\widehat{\mathbb{G}_a}}=\mathrm{RHom}^{\otimes k[\![\beta]\!]}_{\mathrm{PreMF}(M^2,-f\boxplus f)}(\overline{\Delta}_*\mathcal{O}_M,\overline{\Delta}_*\omega_M)$$

which evidently refines that on $\mathbf{HH}^{k(\!(\beta)\!)}_\bullet(\mathrm{MF}(M,f))=\mathbf{HH}^k_\bullet(\mathrm{Perf}(M))^{\mathrm{Tate}}$. We must produce a lift of $[\mathrm{vol}_M]\in\mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M,\Delta_*\omega_M[-m])$ to $\mathrm{SO}(2)\times B\widehat{\mathbb{G}_a}$ invariants.

The mere existence of a lift is actually automatic: Since

$$\mathrm{Map}=\mathrm{Map}_{M^2}(\Delta_*\mathcal{O}_M,\Delta_*\omega_M[-m])=\Omega^\infty\,\mathrm{RHom}_{M^2}(\Delta_*\mathcal{O}_M,\Delta_*\omega_M[-m])$$

with both $\Delta_*\mathcal{O}_M$ and $\Delta_*\omega_M[-m]$ in the heart of the t-structure, this space is *discrete*. Regarding it as a complex in degree 0, it has an action of $\mathrm{SO}(2)\times B\widehat{\mathbb{G}_a}$ such that the map to the whole Hochschild complex is equivariant. But, since it is in degree 0 the action cannot help but be trivializable. So, $[\mathrm{vol}_M]$ admits a lift to fixed points which is unique up to contractible choices; and similarly, $\pi_0\,\mathrm{Map}^{\mathrm{SO}(2)\times B\widehat{\mathbb{G}_a}}=\pi_0\,\mathrm{Map}=\mathrm{Map}$.

As before, it is possible to make the choice naturally (i.e., dependent only on some universal choice). We describe how to do this in the case where M is a smooth variety, so that we can use HKR descriptions: We will produce a lift

$$\mathrm{RHom}_M(\mathcal{O}_M,\omega_M[-m])\longrightarrow\mathrm{RHom}_{\mathrm{PreMF}(M^2,-f\boxplus f)}(\overline{\Delta}_*\mathcal{O}_M,\overline{\Delta}_*\omega_M[-m])^{\mathrm{SO}(2)}$$

of $\overline{\Delta}_*$ in the HKR model of Theorem 6.1.2.5. There's an obvious map

$$\mathrm{R}\Gamma(\Omega^m_M)\to\left(\mathbf{HH}^k_\bullet(\mathrm{Perf}\,M)[-m]\right)^{B\widehat{\mathbb{G}_a}\times\mathrm{SO}(2)}=\mathrm{R}\Gamma\left([\Omega^\bullet_M[-m][\![\beta,u]\!],\beta\cdot(-df\wedge-)+u\cdot d]\right)$$

since Ω^m_M is the degree 0 piece of the complex of sheaves on the right, and this piece has no differentials into it (there's nothing in positive degree) or out of it (both d and $-df\wedge-$ vanish for degree reasons). Again, the map on connective covers can be identified with the identity on $H^0(M,\Omega^m_M)$. □

Remark 6.1.3.5. The Claim is also apparent in a Dolbeault model (over $\mathbb{C}$): If $\mathrm{vol}_M\in$

$\Gamma(\mathcal{A}^{m,0})$ is a holomorphic volume form, it evidently gives rise to a cycle in

$$\left(\mathbf{HH}^k_\bullet(\operatorname{Perf} M)[-m]\right)^{B\widehat{\mathbb{G}_a}\times \mathrm{SO}(2)} = \left[(\oplus_{p,q}\Gamma(\mathcal{A}^{p,q})[p-q-m])\,[\![\beta,u]\!], \overline{\partial} + \beta\cdot(-df\wedge -) + u\cdot\partial\right]$$

Indeed, $\overline{\partial}\operatorname{vol}_M$ vanishes since vol_M is holomorphic, while $\partial \operatorname{vol}_M$ and $-df\wedge\operatorname{vol}_M$ vanish since they would have to be $(m+1,0)$-forms.

6.2 Quadratic bundles

The goal of this section is two-fold:

- We carry out a first class of computation of PreMF, in the spirit of Kapustin-Li: For non-degenerate quadratic bundles over a space, PreMF (with supports along the zero section) admits a description in terms of a $k[\![\beta]\!]$-linear variant of sheaves of Clifford algebras. (Upon inverting β, this recovers a relative form of the computations of Kapustin-Li for matrix factorizations.)

- We use a variant of Theorem 3.2.1.3 to prove a a relative form of Knörrer periodicity for metabolic quadratic bundles, and so re-construct using MF a 2-periodic version of the Clifford invariant on the Witt group. Note that Knörrer-type result is valid only after inverting β.

6.2.1 Metabolic quadratic bundles and relative Knörrer periodicity

Definition 6.2.1.1. A *quadratic bundle* $(\mathscr{Q}, Q)$ over a scheme X is a pair consisting of: a locally free sheaf $\mathscr{Q}$, and a *non-degenerate* symmetric bilinear pairing $Q\colon \mathscr{Q}\otimes_{\mathcal{O}_X}\mathscr{Q}\to\mathcal{O}_X$.[3]

6.2.1.2. We associate to a quadratic bundle $(\mathscr{Q}, Q)$ over X:

- The total space $\mathcal{Q}\to X$, a scheme smooth over X: $\mathcal{Q} = \mathbb{A}(\mathscr{Q}^\vee) \overset{\text{def}}{=} \operatorname{Spec}_X \operatorname{Sym}_{\mathcal{O}_X}\mathscr{Q}^\vee$.

- The quadratic form $q\colon \mathcal{Q}\to\mathbb{A}^1$: defined on points by $q(v) = \frac{1}{2}Q(v\otimes v)$ (or on sheaves, by $\mathcal{O}_X \to \mathscr{Q}^\vee\otimes\mathscr{Q}^\vee\to \operatorname{Sym}^2(\mathscr{Q}^\vee)$).

We will regard $(\mathcal{Q}, q)$ as an LG pair.

Lemma 6.2.1.3. *Let $(\mathscr{Q}, Q)$ be a quadratic bundle over a smooth scheme X, and $(\mathcal{Q}, q)$ the resulting LG pair. Then:*

(i) $\operatorname{crit}(q) = X$*, in particular* 0 *is the only critical value.*

(ii) There is a natural identification $\mathscr{N}_{X/\mathcal{Q}} = \mathscr{Q}$, under which Q corresponds to the Hessian. In particular, q is Morse-Bott.

Proof.

[3] Non-degenerate means that the induced sheaf map $\mathscr{Q}\to\mathscr{Q}^\vee$ is an isomorphism.↑

(i) Follows from the condition that Q is non-degenerate: Working locally, we may suppose $\mathscr{Q} = \oplus_{i=1}^{n} \mathcal{O}_X v_i$, so that $\mathcal{O}_{\mathcal{Q}} = \mathcal{O}_X[x_1, \dots, x_n]$ (x_i dual to v_i) and $q = \frac{1}{2}\sum_{i,j} Q(v_i, v_j) x_i x_j$. Then, $\mathrm{crit}(q)$ is cut out by the equations

$$0 = \frac{dq}{dx_i} = \sum_j Q(v_i, v_j) x_j \qquad \text{for } i = 1, \dots, n$$

$$0 = \xi \cdot dq = \frac{1}{2} \sum_{i,j} \left(\xi \cdot dQ(v_i, v_j)\right) x_i x_j \qquad \text{for } \xi \in TX$$

The first set of equations may be reformulated as the vanishing of the vector $Q(x_1, \dots, x_n)^T$. Since Q was assumed non-degenerate, this cuts out precisely the locus $x_1 = \dots, x_n = 0$, i.e., X. The second set of equations are contained in the ideal generated by the first, i.e., they vanish along X as well.

(ii) The (dual) identification is routine: $\mathscr{N}^{\vee}_{X/\mathcal{Q}} = \mathcal{I}_X|_X = \mathscr{Q}^{\vee} \otimes \mathcal{O}_{\mathcal{Q}}\Big|_X = \mathscr{Q}^{\vee}$. The previous computation in local coordinates shows that

$$\frac{dq}{dx_i dx_j} = Q(v_i, v_j)$$

which, tracing through the identification in this case, proves the claim about the Hessian. □

Remark 6.2.1.4. The formal Morse Lemma tells us that the LG pairs $(\mathcal{Q}, q)$ are (formally locally) representative of LG pairs with Morse-Bott singularities.

6.2.1.5. Suppose X is a smooth variety. Regard $\mathrm{Perf}(X)^{\otimes}$ as a symmetric monoidal ∞-category, and let $\mathrm{Perf}(X)[\![\beta]\!]^{\otimes} = \mathrm{Perf}(X) \otimes_k k[\![\beta]\!]$ (resp., $\mathrm{Perf}(X)(\!(\beta)\!)^{\otimes} = \mathrm{Perf}(X) \otimes_k k(\!(\beta)\!)$) be the associated $k[\![\beta]\!]$- (resp., $k(\!(\beta)\!)$-)linear symmetric monoidal ∞-category. If $\mathcal{C}, \mathcal{D}$ are $\mathrm{Perf}(X)[\![\beta]\!]$- (resp., $\mathrm{Perf}(X)(\!(\beta)\!)$-)module dg-categories, let us denote

$$\mathcal{C} \otimes_{X[\![\beta]\!]} \mathcal{D} \overset{\text{def}}{=} \mathcal{C} \otimes_{\mathrm{Perf}(X)[\![\beta]\!]} \mathcal{D} \qquad \left(\text{resp., } \mathcal{C} \otimes_{X(\!(\beta)\!)} \mathcal{D} \overset{\text{def}}{=} \mathcal{C} \otimes_{\mathrm{Perf}(X)(\!(\beta)\!)} \mathcal{D}\right)$$

6.2.1.6. Earlier in the paper, we noted that Knörrer periodicity could be deduced from our Thom-Sebastiani Theorem together with an explicit computation of matrix factorizations for a rank 2 quadratic form. Part (ii) of the following Theorem provides a globalized version of Knörrer periodicity. Part (i) of the following Theorem provides a relative form of the Thom-Sebastiani Theorem, under an additional hypothesis:

Theorem 6.2.1.7 (Relative Knörrer Periodicity)**.** *Suppose X is a smooth variety, $(\mathscr{Q}, Q)$ is a quadratic bundle over X, and $(\mathcal{Q}, q)$ is the associated LG pair.*

(i) Suppose (Y, f) is a relative LG pair over X: That is Y is a smooth X-scheme equipped with a map f to $\mathbb{A}^1$. For any closed subset $Z \subset f^{-1}(0)$, exterior tensor product induces $\mathrm{Perf}(X)[\![\beta]\!]$*- (resp.,* $\mathrm{Perf}(X)(\!(\beta)\!)$*-)linear equivalences*

$$\mathrm{PreMF}_Z(Y, f) \otimes_{X[\![\beta]\!]} \mathrm{PreMF}_X(\mathcal{Q}, q) \xrightarrow{\sim} \mathrm{PreMF}_{Z \times_X X}(Y \times_X \mathcal{Q}, f \boxplus q)$$

$$\mathrm{MF}_Z(Y, f) \otimes_{X[\![\beta]\!]} \mathrm{MF}(\mathcal{Q}, q) \xrightarrow{\sim} \mathrm{MF}_{Z \times_X X}(Y \times_X \mathcal{Q}, f \boxplus q)$$

(ii) *Suppose* $(\mathscr{Q}, Q)$ *is a metabolic quadratic bundle, i.e., there is a locally free subsheaf* $\mathscr{L} \subset \mathscr{Q}$ *such that* $\mathscr{L} = \mathscr{L}^\perp$ *and* $\mathscr{L}$ *is locally a direct summand (i.e., a subbundle). Let* $\mathcal{L} = \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X} \mathscr{L}^\vee$ *be the total space of* $\mathscr{L}$*. Regard* $\mathcal{L}$ *as a closed subscheme of* $\mathcal{Q}_0$*, so that* $\mathcal{O}_\mathcal{L}$ *is an object of* $\mathrm{DCoh}(\mathcal{Q}_0)$ *and thus of* $\mathrm{MF}(\mathcal{Q}, q)$*. Then, tensoring with* $\mathcal{O}_\mathcal{L}$ *induces an equivalence*

$$\mathrm{Perf}(X)((\beta)) = \mathrm{MF}(X, 0) \longrightarrow \mathrm{MF}(\mathcal{Q}, q).$$

Proof.

(i) The first equivalence follows from the proof of Theorem 3.2.1.3, replacing the reference to Prop. A.2.3.2 with Prop. A.2.4.1 (applied to the second factor, since X is certainly always smooth over X). The second equivalence follows from the first.

(ii) It suffices to prove the Ind-completed version, i.e., that $\mathrm{QC}(X)[\![\beta]\!] \to \mathrm{MF}^\infty(\mathcal{Q}, q)$ is an equivalence. Both sides are étale sheaves (Prop. A.1.3.1) and the functor is evidently local, so that the claim is local. We are thus free to assume that $X = \operatorname{Spec} R$. It now suffices to verify the following two claims:
Claim 1: $\mathcal{O}_\mathcal{L}$ generates $\mathrm{MF}(\mathcal{Q}, q)$ (recall, X is affine).
Note that $\mathcal{L} \supset X = \mathrm{crit}(\mathcal{Q}_0)$, so the inclusion $\mathrm{MF}_\mathcal{L}(\mathcal{Q}, q) \to \mathrm{MF}(\mathcal{Q}, q)$ is an equivalence by Prop. 3.2.1.6. It thus suffices to show that $\mathcal{O}_\mathcal{L}$ generates $\mathrm{DCoh}_\mathcal{L}(\mathcal{Q}_0)$. By Lemma 2.2.0.2, $\mathrm{DCoh}_\mathcal{L}(\mathcal{Q}_0)$ is generated by the image of $i_*\colon \mathrm{DCoh}(\mathcal{L}) \to \mathrm{DCoh}_\mathcal{L}(\mathcal{Q}_0)$ so that it suffices to show that $\mathcal{O}_\mathcal{L}$ generates $\mathrm{DCoh}(\mathcal{L}) = \mathrm{Perf}(\mathcal{L})$. Since X was assumed affine, so is $\mathcal{L}$ and the claim follows by the Hopkins-Neeman-Thomason Theorem.

Claim 2: The natural map $\mathcal{O}_X[\![\beta]\!] \to \mathrm{RHom}^\otimes_{\mathcal{Q}_0}(\mathcal{O}_\mathcal{L}, \mathcal{O}_\mathcal{L})$ becomes an equivalence after $- \otimes_{k[\![\beta]\!]} k((\beta))$.
The claim is local on X, so that we may assume that

- There are trivializations $\mathscr{L} \simeq \oplus_{i=1}^r \mathcal{O}_X \cdot y_i$ and $\mathscr{L}^\vee \simeq \oplus_{i=1}^r \mathcal{O}_X \cdot x_i$.
- $(\mathscr{Q}, Q)$ is not just metabolic, but hyperbolic (see e.g., Bass' work on quadratic forms over rings): i.e., there exists an isomorphism $(\mathscr{Q}, Q) \simeq H(\mathscr{L}) \stackrel{\text{def}}{=} (\mathscr{L} \oplus \mathscr{L}^\vee, Q_H)$ where Q_H is just the natural duality pairing pairing.

In terms of the above local identifications:

$$\mathcal{O}_\mathcal{L} = \mathcal{O}_X[x_1, \ldots, x_r], \quad \mathcal{O}_{\mathcal{Q}_0} = \mathcal{O}_X \begin{bmatrix} x_1, \ldots, x_r, \\ y_1, \ldots, y_r \end{bmatrix} /q, \text{ and } \quad q = \sum_{i=1}^r x_i y_i$$

Writing $\mathcal{O}_{\mathcal{Q}_0} \sim \left(\operatorname{Sym}_{\mathcal{O}_X} \mathscr{Q}^\vee\right)[\epsilon]/d\epsilon = q$, we are led to the following Koszul-Tate resolution of $\mathcal{O}_\mathcal{L}$ over $\mathcal{O}_{\mathcal{Q}_0}$:

$$\begin{aligned}\mathcal{O}_\mathcal{L} &\sim \mathrm{Kos}_{\mathcal{O}_{\mathcal{Q}_0}}\left(\mathscr{L}^\vee \otimes \mathcal{O}_{\mathcal{Q}_0} \xrightarrow{y_i} \mathcal{O}_{\mathcal{Q}_0}\right)\left[\frac{u^k}{k!}\right] / \{du = -e\} \\ &\sim \underbrace{\mathcal{O}_{\mathcal{Q}_0}[\delta_1, \ldots, \delta_r, u^k/k!]}_{\deg \delta_i = +1, \deg u = +2} / \left\{ \begin{matrix} \delta_i^2 = 0 \\ d\delta_i = y_i \\ du = -\sum x_i \delta_i \end{matrix} \right\}\end{aligned}$$

where $e \in \mathrm{Kos}_{\mathcal{O}_{\mathcal{Q}}}(\mathscr{L}^\vee \otimes \mathcal{O}_{\mathcal{Q}} \to \mathcal{O}_{\mathcal{Q}})$ satisfies $d(e) = q$ (in local coordinates, it may be given by the formula above). The natural $\mathcal{O}_X[\![\beta]\!]$ action on this resolution admits the following description: β acts by $\beta = d/du$, while $\mathcal{O}_X$ acts by multiplication. It remains to use the resolution to compute $\mathrm{RHom}^{\otimes}_{\mathcal{Q}_0}(\mathcal{O}_{\mathcal{L}}, \mathcal{O}_{\mathcal{L}})$ as $\mathcal{O}_X[\![\beta]\!]$-module, and show that after inverting β it is a free module on the identity morphism. Dualizing the differentials, we readily compute:

$$\mathrm{RHom}^{\otimes}_{\mathcal{Q}_0}(\mathcal{O}_{\mathcal{L}}, \mathcal{O}_{\mathcal{L}}) = \underbrace{\mathcal{O}_{\mathcal{L}}[\![\beta]\!]\,[\gamma_1, \dots, \gamma_r]}_{\deg \gamma_i = -1, \deg \beta = -2} / \left\{ \begin{array}{c} \gamma_i^2 = 0 \\ d\gamma_i = -x_i \beta \end{array} \right\}$$

where 1 is the identity map. Note that the "y_i" Koszul differentials have gone to zero, and we are left only with the differentials in the "u-direction." Moreover these remaining differentials are all part of potentially truncated "$-x_i$" Koszul complexes. Upon inverting β, the truncation disappears and we obtain a splitting into shifts of a Koszul complexes resolving $\mathcal{O}_X$:

$$\begin{aligned} \mathrm{RHom}^{\otimes}_{\mathcal{Q}_0}(\mathcal{O}_{\mathcal{L}}, \mathcal{O}_{\mathcal{L}}) \otimes_{k[\![s]\!]} k(\!(\beta)\!) &= \mathcal{O}_{\mathcal{L}}(\!(\beta)\!)\,[\gamma_1, \dots, \gamma_r] / \left\{ \begin{array}{c} \gamma_i^2 = 0 \\ d\gamma_i = -x_i t \end{array} \right\} \\ &= \bigoplus_{i \in \mathbb{Z}} t^i \, \mathrm{Kos}_{\mathcal{O}_{\mathcal{L}}} \left(t^{-1} \mathscr{L} \otimes \mathcal{O}_{\mathcal{L}} \xrightarrow{-x_i} \mathcal{O}_{\mathcal{L}} \right) \\ &\simeq \bigoplus_{i \in \mathbb{Z}} t^i \mathcal{O}_X = \mathcal{O}_X(\!(\beta)\!) \end{aligned}$$

□

6.2.2 Witt group and "derived Azumaya algebras"

6.2.2.1. Suppose $(\mathscr{Q}_1, Q_1)$, $(\mathscr{Q}_2, Q_2)$ are quadratic bundles over X, with associated LG pairs $(\mathcal{Q}_1, q_1)$, $(\mathcal{Q}_2, q_2)$. Form the "orthogonal sum" $(\mathscr{Q}_1 \oplus \mathscr{Q}_2, Q_1 \perp Q_2)$; its associated LG pair will be $(\mathcal{Q}_1 \times_X \mathcal{Q}_2, q_1 \boxplus q_2)$.

Define the *Witt semigroup* $W^s(X)$ of X to be the semi-group of (isomorphism classes of) quadratic bundles over X, equipped with orthogonal sum. Define the *Grothendieck-Witt group* $GW(X)$ to be the Grothendieck group of the Witt semigroup. Define the *Witt group* $W(X)$ to be the quotient of $GW(X)$ by the subgroup generated by metabolic quadratic bundles.

Any element of $GW(X)$ may be written in the form $\mathscr{Q}_1 - \mathscr{Q}_2$. Letting $\overline{\mathscr{Q}}_2$ denote $\mathscr{Q}_2$ equipped with the negative quadratic form, we may rewrite

$$\mathscr{Q}_1 - \mathscr{Q}_2 = \left(\mathscr{Q}_1 \perp \overline{\mathscr{Q}}_2\right) - \left(\mathscr{Q}_2 \perp \overline{\mathscr{Q}}_2\right)$$

where now $\mathscr{Q}_2 \perp \overline{\mathscr{Q}}_2$ is *metabolic* (with Lagrangian subspace $\mathscr{L} = \Delta_{\mathscr{Q}_2}$ the diagonal). In particular, $W(X)$ is the quotient semigroup of $W^s(X)$ by the metabolic elements.

Thus Theorem 6.2.1.7 implies

Corollary 6.2.2.2. *The assignment*

$$(\mathscr{Q}, Q) \longrightarrow \mathrm{MF}(\mathcal{Q}, q)$$

takes orthogonal sum of quadratic bundles to tensor product of ∞*-categories over* $\mathrm{Perf}(X)(\!(\beta)\!)$*. It takes isomorphisms to equivalences. It takes metabolic bundles to the tensor unit (i.e.,*

$\mathrm{MF}(X,0) = \mathrm{Perf}(X)((\beta))$). *Therefore, it descends to a group homomorphism*

$$(W(X), \perp) \longrightarrow \left\{ \begin{matrix} \textit{Equivalence classes of invertible} \\ \mathrm{Perf}(X)((\beta))\textit{-linear } \infty\textit{-categories} \end{matrix}, - \otimes_{X((\beta))} - \right\}$$

Remark 6.2.2.3. In the statement of the previous Corollary, "invertible" means in the sense of invertible object for the tensor product $- \otimes_{X((\beta))} -$. The right hand side is thus a 2-periodic version a "derived Azumaya algebra" of Toën [T3].

6.2.3 Relation to Clifford bundles

6.2.3.1. At this point (if not earlier), the conscientious reader should object: There's a more down-to-Earth construction of (usual) Azumaya algebras out of elements in the Witt group, by taking the bundle of Clifford algebras associated to $\mathscr{Q}$. The following Theorem explains this. Since it's proof is independent of the above, we could presumably have proven part (ii) of Theorem 6.2.1.7 in the world of Clifford algebras.[4]

6.2.3.2. Suppose $(\mathscr{Q}, Q)$ is a quadratic bundle on a scheme X. Then, $\mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q})$ is the following sheaf of $\mathbb{Z}/2$-graded algebras

$$\mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q})_{\mathbb{Z}/2} \stackrel{\text{def}}{=} \mathcal{O}_X\langle \mathscr{Q} \rangle / \left\{ v^2 = -Q(v,v) \right\}$$

where $\mathscr{Q}$ is in odd degree, and v denotes a section of $\mathscr{Q}$.

6.2.3.3. In this paper, it has been our convention to replace $\mathbb{Z}/2$-graded objects with $\mathbb{Z}$-graded objects over $k((\beta))$, $\deg \beta = -2$. Under this equivalence, the above sheaf of algebras goes to

$$\mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q}) \stackrel{\text{def}}{=} \mathcal{O}_X((\beta))\langle \mathscr{Q} \rangle / \left\{ v^2 = -Q(v,v)\beta \right\}$$

where $\mathscr{Q}$ is in degree -1, and β is in degree -2. There is also a $k[\![\beta]\!]$-linear version:

$$\mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q}) \stackrel{\text{def}}{=} \mathcal{O}_X[\![\beta]\!]\langle \mathscr{Q} \rangle / \left\{ v^2 = -Q(v,v)\beta \right\}$$

Theorem 6.2.3.4 (Relative Kapustin-Li). *Suppose X is a smooth scheme, $(\mathscr{Q}, Q)$ a quadratic bundle on X, and $(\mathcal{Q}, q)$ the associated LG pair. Then, the structure sheaf $\mathcal{O}_X$ induces a natural equivalence of $k[\![\beta]\!]$-linear dg-categories*

$$\mathrm{PreMF}^{\infty}_X(\mathcal{Q}, q) \simeq \mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q})\text{-mod}(\mathrm{QC}(X))$$

and of $k((\beta))$-linear dg-categories

$$\mathrm{MF}^{\infty}(\mathcal{Q}, q) \simeq \mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q})\text{-mod}(\mathrm{QC}(X)) = \mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q})_{\mathbb{Z}/2}\text{-mod}_{dg\mathbb{Z}/2}(\mathrm{QC}(X))$$

Remark 6.2.3.5. Before giving a complete proof of the Theorem, we should point that it is in essence a straightforward computation of a relative Koszul dual over $\mathcal{O}_X$ (with a little extra book-keeping for the $k[\![\beta]\!]$-action). In local coordinates, it is saying that the Koszul

[4] Though I'm not aware of the desired $\mathbb{Z}/2$-graded Morita equivalence appearing in the literature in the metabolic case. If X is affine, then any metabolic bundle is hyperbolic and the Morita equivalence is well-known and visibly $\mathbb{Z}/2$-graded. [KO] shows that the Brauer class does vanish in the metabolic case, but that it is is not necessarily the case that $\mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q}) \simeq \mathrm{End}_{\mathcal{O}_X}(\bigwedge^* \mathscr{L})$ as one might naively guess.↑

dual of the dg-algebra

$$\mathcal{O}_{\mathcal{Q}_0} \sim \mathcal{O}_X[x_1, \ldots, x_n][\epsilon]/d\epsilon = q$$

is $\mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q})$ viewed as a filtered algebra (depending on the differential above), having associated graded

$$\mathcal{O}_X[\delta_1, \ldots, \delta_n][t]$$

Proof. We first outline our plan of proof: We first use descent to reduce to the affine case, and note that (locally on X) $\mathrm{PreMF}_X(\mathcal{Q}, q)$ (and $\mathrm{MF}(\mathcal{Q}, q)$) are generated by $\mathcal{O}_X$. It thus suffices to identify $\mathrm{RHom}^{\otimes}_{\mathcal{Q}_0/X}(\mathcal{O}_X, \mathcal{O}_X)$ with $\mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q})$ and analogously for MF. In Step 2, we will explicitly construct a resolution on which we can see the $k[\![\beta]\!]$-action and use this to compute the underlying complex of the endomorphisms in Step 3. Finally, to identify the algebra structure we explicitly describe the Clifford action on this resolution in Step 4.

Step 1: Identifying the generator.
Note that both sides are étale sheaves on X by Section A.1. In the following steps, we will identify $\mathrm{RHom}^{\otimes X}_{\mathrm{PreMF}(\mathcal{Q},Q)}(\mathcal{O}_X \mathcal{O}_X) \simeq \mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q})$ as sheaves of algebras, i.e., objects in $\mathbf{Alg}(\mathrm{QC}(X))$; note that this implies the analogous identifies on $- \otimes_{k[\![\beta]\!]} k(\!(\beta)\!)$. Then, the functor in question will be $\mathrm{RHom}^{\otimes X}_{\mathrm{PreMF}^{\infty}(\mathcal{Q},Q)}(\mathcal{O}_X, -)$ (resp., $\mathrm{RHom}^{\otimes X}_{\mathrm{MF}^{\infty}}$), factored through $\mathrm{RHom}^{\otimes X}_{\mathrm{PreMF}^{\infty}(\mathcal{Q},Q)}(\mathcal{O}_X, -)$-modules (resp., MF^{∞}); in particular, the functor is local on X by Lemma A.1.1.1. To complete the proof it then suffices to show that $\mathrm{RHom}^{\otimes X}_{\mathrm{PreMF}^{\infty}(\mathcal{Q},Q)}(\mathcal{O}_X, -)$ (resp. MF) is an equivalence locally on X. To do this, it is enough by Morita theory to note that locally on X $\mathcal{O}_X$ generates $\mathrm{PreMF}_X(\mathcal{Q}, Q)$ by Lemma 2.2.0.2; and $\mathcal{O}_X$ generates $\mathrm{MF}(\mathcal{Q}, Q)$ by the preceding, Lemma 6.2.1.3, and Prop. 3.2.1.6.

Step 2: Constructing the resolution.
Let $j \colon \mathcal{Q}_0 \to \mathcal{Q}$ be the inclusion. For $\mathscr{F} \in \mathrm{DCoh}(\mathcal{Q}_0)$, recall the functorial resolution of Example 3.1.1.11

$$\mathscr{F} \xleftarrow{\sim} \mathrm{Tot}^{\oplus}\left\{ j^* j_* \mathscr{F} \xleftarrow{B} \Sigma j^* j_* \mathscr{F} \xleftarrow{B} \Sigma^2 j^* j_* \mathscr{F} \leftarrow \cdots \right\}$$

We begin with the Koszul resolution of $j_*(i_* \mathcal{O}_X)$ over $\mathcal{O}_{\mathcal{Q}}$, which we will think of in two ways:

$$j_*(i_*\mathcal{O}_X) \xleftarrow{\sim} \mathrm{Kos}_{\mathcal{O}_{\mathcal{Q}}}\left(m \colon \mathscr{Q}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Q}} \to \mathcal{O}_{\mathcal{Q}}\right) = \left(\Omega^{\bullet}_{\mathcal{Q}/X}, i_E = \sum_i x_i \partial_{x_i}\right)$$

where m is the "multiplication" map (recall, $\mathcal{O}_{\mathcal{Q}} = \mathrm{Sym}_{\mathcal{O}_X} \mathscr{Q}^{\vee}$). Here we have identified $\mathscr{Q}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Q}} = \Omega^1_{\mathcal{Q}/X}$, and so have identified the Koszul complex with the relative differential forms. The multiplication map gives rise to a differential on this, which can be described as contraction i_E with an "Euler vector field" $E = \sum_i x_i \frac{\partial}{\partial x_i}$. Pulling back to $\mathcal{Q}_0$ we obtain

$$j^* j_*(i_*\mathcal{O}_X) \xleftarrow{\sim} \mathrm{Kos}_{\mathcal{O}_{\mathcal{Q}_0}}\left(m \colon \mathscr{Q}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Q}_0} \to \mathcal{O}_{\mathcal{Q}_0}\right) = \left(j^* \Omega^{\bullet}_{\mathcal{Q}/X}, i_E\right)$$

It remains to compute the map $B \colon \Sigma j^* j_* \mathscr{F} \to j^* j_* \mathscr{F}$ in these terms: It is (the restriction to $\mathcal{O}_{\mathcal{Q}_0}$ of) left-multiplication by the section

$$\frac{dq}{2} \in \Gamma(\mathcal{Q}, \Omega^1_{\mathcal{Q}/X}) \qquad \text{or} \qquad \frac{1}{2} \sum_i x_i \otimes \frac{dq}{dx_i} \in \Gamma(\mathcal{Q}, \mathscr{Q}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Q}})$$

This satisfies $i_E(Bx) = q\cdot x - B i_E(x)$ since q is homogeneous of degree 2 so that $i_E(dq/2) = q$.

Putting this together, we obtain the following resolution for $i_*\mathcal{O}_X$ as $\mathcal{O}_{\mathcal{Q}_0}$-module. For convenience, we follow the Tate convention of writing Koszul-type complexes in terms of graded-commutative (divided-power) algebras:

$$i_*\mathcal{O}_X \xleftarrow{\sim} (j^*)\left(\mathcal{O}_{\mathcal{Q}_0}\left[\underbrace{\Sigma(\mathscr{Q}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Q}_0})}_{\deg=+1}\right]\left[\underbrace{u^k/k!}_{\deg u=+2}\right], \begin{array}{c} d(x\otimes a) = xa \\ d(u) = dq/2 \end{array}\right) = \left[\left(j^*\Omega^\bullet_{\mathcal{Q}/X}\right)[u^k/k!], i_E + \frac{dq}{2}\frac{\partial}{\partial u}\right]$$

Step 3: Identifying the underlying complex.
Using the previous complex, we readily compute that (as object in $\mathrm{QC}(X)$, ignoring the algebra structure)

$$\begin{aligned}
\mathcal{RH}om^{\otimes_X}_{\mathcal{Q}_0}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) &= \mathcal{RH}om^{\otimes_X}_{\mathcal{Q}_0}\left(\left[j^*\Omega^\bullet_{\mathcal{Q}/X}[u^k/k!], m + (dq/2)\cdot\partial/\partial u\right], i_*\mathcal{O}_X\right)\\
&= \mathcal{RH}om_X\left(\left[i^*j^*\Omega^\bullet_{\mathcal{Q}/X}[u^m/m!], 0\right], \mathcal{O}_X\right)\\
&= \mathcal{O}_X[\![\beta]\!]\left[\underbrace{\mathscr{Q}^\vee[-1]}_{\deg=-1}\right]
\end{aligned}$$

Indeed, $\mathrm{RHom}^\otimes$ takes the hocolimit (i.e., $\mathrm{Tot}^\oplus$) in the first variable to a holim (i.e., Tot^Π), and all the differentials vanish.

Step 4: Producing the algebra map.
We wish to produce a map of (sheaves of dg) algebras

$$\phi\colon\ \mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q}) \longrightarrow \mathrm{RHom}^\otimes_{\mathcal{Q}_0/X}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$$

To construct the map, we make $\mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q})$ act on our explicit resolution: $\mathcal{O}_X$ acts via $\mathcal{O}_X \to \mathcal{O}_{\mathcal{Q}_0}$; β acts by d/du (i.e., shifting the resolution in the u). It remains to describe the action for $v \in \mathscr{Q} = T_{\mathcal{Q}/X}$, and show that it satisfies the Clifford relations and (anti-)commutes with the differentials. We define the action of $v \in \Gamma(T_{\mathcal{Q}/X})$ by contracting i_v where we imagine u as standing in for the Hessian tensor; explicitly:

- On the Koszul piece, $\Omega^\bullet_{\mathcal{Q}/X}$, v acts by contraction i_v.
- On u, v acts by taking it to the contraction of the Hessian of q (i.e., Q) by v:

$$v\cdot u = i_v\,\mathrm{Hess}(q)\left(= i_v\left(\sum_{i,j} dx_i dx_j Q(v_i, v_j)\right) = \sum_i dx_i \frac{dq}{dx_i dv}\right)$$

- We extend by requiring the action to be by derivations

$$v\cdot\left(\omega\frac{u^k}{k!}\right) = i_v(\omega)\cdot\frac{u^k}{k!} + (-1)^{|\omega|}\left(\omega\wedge(i_v\,\mathrm{Hess}(q))\right)\frac{u^{k-1}}{(k-1)!}$$

  and linearity.

By explicit computation, this satisfies the Clifford relations

$$\begin{aligned}
v \cdot \left(v \cdot (\omega \frac{u^k}{k!})\right) &= v \cdot \left(i_v(\omega) \cdot \frac{u^k}{k!} + (-1)^{|\omega|}\omega \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!}\right) \\
&= (-1)^{|\omega|-1} i_v(\omega) \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!} + (-1)^{|\omega|} i_v(\omega) \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!} \\
&+ (-1)^{|\omega|+|\omega|-1} \omega \wedge i_v(i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!} \\
&= -\omega \wedge Q(v,v) \frac{u^{k-1}}{(k-1)!} = (-Q(v,v)\beta) \cdot \left(\omega \frac{u^k}{k!}\right)
\end{aligned}$$

and (anti-)commutes with the differential

$$\begin{aligned}
v \cdot d\left(\omega \frac{u^k}{k!}\right) &= v \cdot \left(i_E(\omega) \frac{u^k}{k!} + \left(\frac{dq}{2} \wedge \omega\right) \frac{u^{k-1}}{(k-1)!}\right) \\
&= i_v(i_E(\omega)) \frac{u^k}{k!} + (-1)^{|\omega|-1} i_E(\omega) \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!} \\
&+ i_v(\frac{dq}{2} \wedge \omega) \frac{u^{k-1}}{(k-1)!} + (-1)^{|\omega|+1} (dq/2 \wedge \omega \wedge (i_v \operatorname{Hess}(q))) \frac{u^{k-2}}{(k-2)!}
\end{aligned}$$

while

$$\begin{aligned}
d\left(v \cdot (\omega \frac{u^k}{k!})\right) &= d\left(i_v(\omega) \frac{u^k}{k!} + (-1)^{|\omega|} \omega \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-1}}{(k-1)!}\right) \\
&= i_E(i_v(\omega)) \frac{u^k}{k!} + (-1)^{|\omega|} i_E(\omega \wedge (i_v \operatorname{Hess}(q))) \frac{u^{k-1}}{(k-1)!} \\
&+ \frac{dq}{2} \wedge i_v(\omega) \frac{u^{k-1}}{(k-1)!} + (-1)^{|\omega|} \frac{dq}{2} \wedge \omega \wedge (i_v \operatorname{Hess}(q)) \frac{u^{k-2}}{(k-2)!}
\end{aligned}$$

Step 5: Checking equivalence.
We now verify that the algebra map ϕ of Step 4 induces an equivalence on underlying complexes, by using the description of the underlying complex given in Step 3.

The first observation is that β goes to β: More precisely, the isomorphism of Step 3 is in fact an isomorphism of $\mathcal{O}_X[\![\beta]\!]$-modules; in Step 3, β^k was dual to $u^k/k!$, which is compatible with β acting by d/du. Thus, ϕ is a map of locally free $\mathcal{O}_X[\![\beta]\!]$-modules of the same rank, and it suffices to work locally and match up generators. This is straightforward. □

Remark 6.2.3.6. Sheaves of Clifford algebras and quadratic bundles also appear in Kuznetsov's homological projective duality ([K]). The relationship of those results to the previous Theorem can be loosely summarized as a relative version of the *LG/CY correspondence* ([O3]): Let $\mathbb{P}(\mathcal{Q}_0) \subset \mathbb{P}(\mathscr{Q}^\vee)$ denote the bundle of projective quadrics associated to $(\mathscr{Q}, Q)$. Then, the LG/CY correspondence asserts (assume $\dim \mathscr{Q} \geq 2$) the existence of a fully-faithful functor $\operatorname{DSing}^{\mathrm{gr}} \mathcal{Q}_0 \hookrightarrow \operatorname{DCoh} \mathbb{P}(\mathcal{Q}_0)$.[5]

[5] In general, the picture is a little delicate due to the interaction of "$\mathbb{G}_m$-weight gradings" with duality: The direction of the functor depends on some numerology. Roughly, it is an equivalence if the projective zero-locus is relative Calabi-Yau, fully-faithful in the direction indicated if it is Fano, and fully-faithful the other way if it is of general type.↑

6.2.3.7. Kuznetsov's $\mathfrak{B}$ is our $\mathrm{PreCliff}_{\mathcal{O}_X}(\mathscr{Q}, Q)$ viewed as a weight graded, non differential by formality, algebra. Kuznetsov's $\mathfrak{B}_0$ (the even part of $\mathfrak{B}$) is $(\mathrm{Cliff}_{\mathcal{O}_X}(\mathscr{Q}, Q))_0$ in our notation (the subscript denotes taking weight zero part), where this is regarded as an ungraded, non differential by formality, algebra. Kuznetsov's constructs ($n \geq 2$) a semiorthogonal decomposition of $\mathrm{DCoh}\,\mathbb{P}(\mathcal{Q}_0)$ whose first term is $\mathrm{DCoh}\,\mathfrak{B}_0\text{-mod}(\mathrm{QC}(X))$, which may be identified with $\mathrm{DSing}^{\mathrm{gr}}\,\mathcal{Q}_0$.

Chapter 7

Adjoint Actions for E_2-algebras and the Hochschild Package of Matrix Factorizations

7.1 Introduction

Suppose (M, f) is an LG pair and assume for simplicity that 0 is the only critical value. In chapter 5, we saw that there was a $B\widehat{\mathbb{G}_a}$-action on $\mathrm{DCoh}(M)$ for which $\mathrm{DCoh}(M)^{B\widehat{\mathbb{G}_a}} = \mathrm{PreMF}(M, f)$ as $k[\![\beta]\!]$-linear categories and $\mathrm{DCoh}(M)^{\mathrm{Tate}} = \mathrm{MF}(M, f)$ as $k(\!(\beta)\!)$-linear categories. By Prop. 5.3.3.11, this $B\widehat{\mathbb{G}_a}$-action induces a $B\widehat{\mathbb{G}_a}$-action on the Hochschild invariants of $\mathrm{DCoh}(M)$. In chapter 6, we saw that the Hochschild invariants

$$\begin{aligned}&\Big(\mathbf{HH}^{\bullet}_{k(\!(\beta)\!)}(\mathrm{MF}(M, F)), \mathbf{HH}_{\bullet}^{k(\!(\beta)\!)}(\mathrm{MF}(M, f))\Big)\\ &\simeq \Big(\mathbf{HH}^{\bullet}_{k}(\mathrm{DCoh}(M)), \mathbf{HH}_{\bullet}^{k}(\mathrm{DCoh}(M))\Big)^{\mathrm{Tate}} \in \mathrm{E}_2^{calc}\text{-alg}(k(\!(\beta)\!)\text{-mod})\end{aligned}$$

with their various higher algebraic structures could be computed as the Tate construction for this $B\widehat{\mathbb{G}_a}$ action on the Hochschild invariants of M.

The goal of the present chapter is to leverage this to actually compute these higher structures. Our approach will be two-fold:

– First, we will show that the induced $B\widehat{\mathbb{G}_a}$-action on $\mathbf{HH}^{\bullet}(M) \in \mathrm{E}_2$-alg (resp., $\mathbf{HH}^{\bullet}(M)[+1] \in$ Lie-alg, etc.) depends only on: the E_2- (resp., Lie-, etc.) algebra itself; and the Lie map $k[+1] \to \mathbf{HH}^{\bullet}(M)[+1]$ encoding f.

– Second, we will use a formality theorem of Dolgushev-Tsygan-Tamarkin to get a handle on the E_2-(resp., Lie-, etc.) algebra stucture on $\mathbf{HH}^{\bullet}(M)$ (etc.).

7.1.1 Classical Motivation

Suppose $c \in \mathcal{C}$ and set $\mathcal{A} = \mathrm{End}_{\mathcal{C}}(c)$. Let $\mathcal{A}^{\mathrm{Lie}}$ denote the underlying Lie algebra of the associative algebra $\mathcal{A}$. It is well-known that $\mathcal{A}^{\mathrm{Lie}}$ controls the deformation theory of the object $c \in \mathcal{C}$. Given a deformation $\widetilde{c}$ of $c \in \mathcal{C}$, one may form its endormphisms $\mathrm{End}_{\mathcal{C}}(\widetilde{c})$ to obtain a deformation of $\mathcal{A}$. More precisely, the formation of endomorphisms gives rise to a map of (pre-)formal moduli problems

$$\mathrm{End}(-)\colon \mathcal{C}^{\wedge}_{c} \overset{\text{def}}{=} \left\{ R \in \mathbf{DArt}_k \mapsto \left\{ \begin{array}{c} \widetilde{c} \in \mathcal{C} \otimes_k R \\ \text{with an equivalences } \widetilde{c} \otimes_R k \simeq c \in \mathcal{C} \end{array} \right\} \right\}$$
$$\longrightarrow \mathbf{Alg}^{\wedge}_{\mathcal{A}} \overset{\text{def}}{=} \left\{ R \in \mathbf{DArt}_k \mapsto \left\{ \begin{array}{c} \widetilde{\mathcal{A}} \in \mathbf{Alg}(R) \\ \text{with an equivalence } \widetilde{\mathcal{A}} \otimes_R k \simeq \mathcal{A} \end{array} \right\} \right\}$$

and thus a map of Lie algebras $\mathcal{A}^{\mathrm{Lie}} \to \mathrm{Der}(\mathcal{A})$.

Our starting point is the following simple observation: Though the above construction proceded via $c \in \mathcal{C}$, this map of Lie algebras in fact depends only on the algebra $\mathcal{A}$ and not its realization as $\mathrm{End}_{\mathcal{C}}(c)$. It is nothing more than the *adjoint action* of the $\mathcal{A}^{\mathrm{Lie}}$ on $\mathcal{A}$, which one checks are by algebra derivations. This is the infinitesimal analog of a familiar discrete fact: The group of units $\mathcal{A}^{\times}$ acts on $\mathcal{A}$ by conjugation, and this is an action by algebra automorphisms.

7.1.2 Higher analogs of adjoint actions

The first result for this chapter will be a higher analog of the above:

Theorem 7.1.2.1. *Suppose* $\mathcal{C} \in \mathbf{dgcat}^{\mathrm{idm}}$.

– *Let* $\mathcal{A} = \mathbf{HH}^{\bullet}(\mathcal{C}) \in \mathrm{E}_2\text{-alg}$. *The map of deformation problems*

$$\mathbf{HH}^{\bullet}(-)\colon \mathbf{dgcat}^{\wedge}_{\mathcal{C}} \longrightarrow \mathrm{E}_2\text{-alg}^{\wedge}_{\mathcal{A}}$$

gives rise to a map of Lie algebras

$$\mathbf{HH}^{\bullet}(\mathcal{C})[+1]^{\mathrm{Lie}} \longrightarrow \mathrm{Der}^{\mathrm{E}_2}(\mathcal{A})$$

This map may be naturally identified, up to homotopy, with the E_2 adjoint action *(see §7.4.3). In particular, it depends only on the* E_2*-algebra* $\mathcal{A}$ *and not on the category* $\mathcal{C}$*.*

– *Let* $\mathcal{A} = (\mathbf{HH}^{\bullet}(\mathcal{C}), \mathbf{HH}_{\bullet}(\mathcal{C})) \in \mathrm{E}_2^{calc}\text{-alg}$. *The map of deformation problems*

$$(\mathbf{HH}^{\bullet}(-), \mathbf{HH}_{\bullet}(-))\colon \mathbf{dgcat}^{\wedge}_{\mathcal{C}} \longrightarrow \mathrm{E}_2^{calc}\text{-alg}^{\wedge}_{\mathcal{A}}$$

gives rise to a map of Lie algebras

$$\mathbf{HH}^{\bullet}(\mathcal{C})[+1]^{\mathrm{Lie}} \longrightarrow \mathrm{Der}^{\mathrm{E}_2^{\mathrm{calc}}}(\mathcal{A})$$

This map may be naturally identified, up to homotopy, with the E_2^{calc} adjoint action *(see §7.4.3). In particular, it depends only on the* E_2^{calc}*-algebra* $\mathcal{A}$ *and not on the category* $\mathcal{C}$*.*

Proof. See §7.4.5. □

Note that (i) is included above only as a more familiar sounding orientation for (ii), which is a strict generalization of it. More generally, Section 7.3 will discuss the discrete analog of adjoint actions and the above result for all E_n, $n \geq 1$. Then, Section 7.4 will discuss the infinitesimal results–e.g., the above Theorem–again for all E_n, $n \geq 1$.

In practice, E_2-algebras can be somewhat involved to compute with. In contrast, computations with Lie-algebras tend to be easier. Furthermore, there are formality results giving computationally convenient descriptions of the Lie algebra $\mathbf{HH}^\bullet(\mathcal{C})[+1]^{\mathrm{Lie}}$ in cases of interest. For this reason, we will find it convenient to have the following Lie-variant of the above. It says that if one only wants to Lie-type output one only needs Lie-type input:

Theorem 7.1.2.2. *Suppose* $\mathcal{C} \in \mathbf{dgcat}^{\mathrm{idm}}$.

- *Let* $\mathcal{A} = \mathbf{HH}^\bullet(\mathcal{C})[+1]^{\mathrm{Lie}} \in \text{Lie-alg}$. *The map of deformation problems*

$$\mathbf{HH}^\bullet(-)[+1]^{\mathrm{Lie}} \colon \mathbf{dgcat}_{\widehat{\mathcal{C}}} \longrightarrow \text{Lie-alg}_{\widehat{\mathcal{A}}}$$

  gives rise to a map of Lie algebras

$$\mathbf{HH}^\bullet(\mathcal{C})[+1]^{\mathrm{Lie}} \longrightarrow \mathrm{Der}^{\mathrm{Lie}}(\mathcal{A})$$

  This map may be naturally identified, up to homotopy, with the Lie adjoint action. *In particular, it depends only on the* Lie-*algebra* $\mathcal{A}$ *and not on the category* $\mathcal{C}$.

- *Let* $\mathcal{A} = (\mathbf{HH}^\bullet(\mathcal{C}), \mathbf{HH}_\bullet(\mathcal{C})) \in \mathrm{Lie}[+1]^+_\delta\text{-alg}$ *(see §7.2.1). The map of deformation problems*

$$(\mathbf{HH}^\bullet(-), \mathbf{HH}_\bullet(-)) \colon \mathbf{dgcat}_{\widehat{\mathcal{C}}} \longrightarrow \mathrm{Lie}[+1]^+_\delta\text{-alg}_{\widehat{\mathcal{A}}}$$

  gives rise to a map of Lie algebras

$$\mathbf{HH}^\bullet(\mathcal{C})[+1]^{\mathrm{Lie}} \longrightarrow \mathrm{Der}^{\mathrm{Lie}[+1]^+_\delta\text{-alg}}(\mathcal{A})$$

  This map may be naturally identified, up to homotopy, with the $\mathrm{Lie}[+1]^+_\delta$ adjoint action *(see §7.4.2). In particular, it depends only on the* $\mathrm{Lie}[+1]^+_\delta$*-algebra* $\mathcal{A}$ *and not on the category* $\mathcal{C}$.

Proof. Follows from Theorem 7.1.2.1 and a compatibility between E_2 and Lie adjoint actions Theorem 7.5.4.1. □

7.1.3 Comparison of E_n and P_n adjoint actions

Like Lie algebras, computations with P_2-algebras tend to be tractable. Choosing a Drinfeld associator Φ, one obtains an equivalence of filtered operads $DQ_\Phi \colon P_2 \simeq \mathrm{E}_2$ (resp., $DQ_\Phi \colon \mathrm{Calc}_2 \simeq \mathrm{E}_2^{calc}$) and so a universal, if complicated, way of reducing computations on E_2 algebras to computations on P_2-algerbas. In cases of interest, formality results give computationally convenient descriptions of the P_2-algebras $DQ_\Phi(\mathbf{HH}^\bullet(\mathcal{C}))$ (resp., $DQ_\Phi(\mathbf{HH}^\bullet(\mathcal{C}), \mathbf{HH}_\bullet(\mathcal{C}))$):

Theorem 7.1.3.1 (Dolgushev-Tsygan-Tamarkin [DTT])**.** *Suppose M is a smooth scheme and let*

$$\mathcal{A} = (\mathbf{HH}^\bullet(M), \mathbf{HH}_\bullet(M)) \in \mathrm{E}_2^{calc}\text{-alg}^{filt}$$

$$\mathrm{gr}\mathcal{A} = \big(\mathrm{R}\Gamma(M, \mathrm{Sym}_{\mathcal{O}_M} T_M[-1]), \mathrm{R}\Gamma(M, \mathrm{Sym}_{\mathcal{O}_M} \Omega_M[+1])\big) \in \mathrm{Calc}_2\text{-alg}^{gr}$$

so that both $DQ_\Phi(\mathcal{A})$ and $\mathscr{G}$ can be regarded as filtered Calc_2*-algebras with associated graded naturally identified with the graded P_2-algebra $\mathscr{G}$. (The filtration and grading are induced, by taking* $\mathrm{R}\Gamma$*, from the sheaf-theoretic Postnikov filtration/its associated graded.) Then, there is a "formality" equivalence: That is, an equivalence $DQ_\Phi(\mathcal{A}) \simeq \mathscr{G}$ of filtered* Calc_2-alg *such that the associated graded of this equivalence may is the identity on $\mathscr{G}$ as graded* P_2-alg.

Just like E_2-algebras (resp., E_2^{calc}-algebras), P_2-algebras (resp., Calc_2-algebras) have evident underlying (shifted) Lie algebras and an evident adjoint action construction (Section 7.4). It is natural to ask how the two are related. It is straightforward to check that one is the associated graded of the other, in the sense that:

Proposition 7.1.3.2. *Suppose $\mathcal{A} = F_\bullet\mathcal{A}$ is a filtered algebra over the filtered operad E_2 (resp., E_2^{calc})–i.e., we work with algebras/operads in filtered complexes. Let $\mathrm{gr}\mathcal{A} = \mathrm{gr}F_\bullet\mathcal{A}$ be its associated graded. It is a graded algebra over the graded operad P_2 (resp., Calc_2)–i.e., we work with algebras/operads in graded complexes. Then:*

- $(\mathrm{gr}\mathcal{A})[+1]^{\mathrm{Lie}} = \mathrm{gr}(\mathcal{A}[+1])^{\mathrm{Lie}}$ *as Lie algebras in graded complexes (with Lie bracket in grading 1). The adjoint action construction, in graded complexes, produces $(\mathrm{gr}\mathcal{A})^{\mathrm{ad}} \in P_2\text{-alg}((\mathrm{gr}\mathcal{A})[+1]^{\mathrm{Lie}}\mathrm{grmod})$.*
- $\mathcal{A}[+1]^{\mathrm{Lie}}$ *is a Lie algebra in filtered complexes (with the Lie bracket in filtration 1), and one can consider the symmetric monoidal category $\mathcal{A}[+1]^{\mathrm{Lie}}\mathrm{filtmod}$ of filtered $\mathcal{A}[+1]^{\mathrm{Lie}}$-modules. The adjoint action construction, in filtered complexes, produces $\mathcal{A}^{\mathrm{ad}} \in \mathrm{E}_2\text{-alg}(\mathcal{A}[+1]^{\mathrm{Lie}}\mathrm{filtmod})$. Taking associated gradeds, one obtains $\mathrm{gr}(\mathcal{A}^{\mathrm{ad}}) \in P_2\text{-alg}(\mathrm{gr}(\mathcal{A}[+1]^{\mathrm{Lie}})\mathrm{grmod})$.*
- *There is a natural equivalence $(\mathrm{gr}\mathcal{A})[+1]^{\mathrm{Lie}} \simeq \mathrm{gr}(\mathcal{A}[+1]^{\mathrm{Lie}})$. Identifying graded module categories using it, there is a natural equivalence $(\mathrm{gr}\mathcal{A})^{\mathrm{ad}} = \mathrm{gr}(\mathcal{A}^{\mathrm{ad}})$.*

Proof. See §7.5.3 for a hands-on demonstration. □

One could also ask about compatibility with quantization ("DQ" stands for de-quantization), starting with the remark that there is a natural equivalence $\mathcal{A}^{\mathrm{Lie}} \simeq DQ_\Phi(\mathcal{A})^{\mathrm{Lie}}$. However, we are – somewhat embarassingly! – not at present able to verify the following conjecture about the compatibility of these constructions under quantization!

Conjecture 7.1.3.3. *Suppose $\mathcal{A} \in \mathrm{E}_2\text{-alg}$ (resp., E_2^{calc}). Then, the following diagram of Lie algebras is naturally commutative up to homotopy*

$$\begin{array}{ccc} \mathcal{A}[+1]^{\mathrm{Lie}} & \xrightarrow{\mathrm{E}_2\ ad.} & \mathrm{Der}^{\mathrm{E}_2}(\mathcal{A}) \\ \downarrow\wr & & \downarrow\wr\, DQ_\Phi \\ DQ_\Phi(\mathcal{A})[+1]^{\mathrm{Lie}} & \xrightarrow{P_2\ ad.} & \mathrm{Der}^{\mathrm{P}_2}(\mathcal{A}) \end{array}$$

As a workaround, we note that for many of our applications it will suffice to have the following much weaker result:

Proposition 7.1.3.4. *Suppose M is a smooth scheme and let $\mathcal{A} = \underline{\mathbf{HH}}^\bullet(M) \in \mathrm{E}_2\text{-alg}(\mathrm{QC}(M))$ be the sheaf of Hochschild cohomologies and $\mathscr{G} = \mathrm{gr}\mathcal{A} = \mathrm{Sym}_{\mathcal{O}_M} T_M[-1] \in P_2\text{-alg}(\mathrm{QC}(M))$ its associated graded sheaf. Let $C^\bullet_{\mathrm{DR}}(M) \in \mathrm{Shv}(M)$ denote the sheaf of de Rham complexes of M. Then,*

(i) There is a fiber sequence of sheaves of Lie algebras on M $C^\bullet_{\mathrm{DR}}(M)[+1] \to \mathscr{G}[+1] \overset{ad}{\to} \mathrm{Der}^{\mathrm{P}_2}(\mathscr{G})$, where $C^\bullet_{\mathrm{DR}}(M)[+1]$ is viewed as an abelian Lie algebra.

(ii) $\pi_i\mathrm{Der}^{\mathrm{P}_2}(\mathscr{G}) = 0$ for all $i > 1$, and $\pi_1\mathrm{Der}^{\mathrm{P}_2}(\mathscr{G})$ can be identified with the sheaf of closed 1-forms on M.

(iii) The following diagram (of sheaves of Lie algebras homologically concentrated in degree +1) is commutative

$$\begin{array}{ccc}
\tau_{\geq 1}\left(\mathcal{A}[+1]^{\mathrm{Lie}}\right) & \xrightarrow{\mathrm{E}_2\ ad.} & \tau_{\geq 1}\left(\mathrm{Der}^{\mathrm{E}_2}(\mathcal{A})\right) \\
\downarrow\wr & & \downarrow\wr\, DQ_\Phi \\
\tau_{\geq 1}\left(DQ_\Phi(\mathcal{A})[+1]^{\mathrm{Lie}}\right) & \xrightarrow{P_2\ ad.} & \tau_{\geq 1}\left(\mathrm{Der}^{\mathrm{P}_2}(\mathcal{A})\right)
\end{array}$$

(iv) The following diagram (of Lie algebras homologically concentrated in degrees ≥ 1) is commutative

$$\begin{array}{ccccc}
\tau_{\geq 1}\left(\mathbf{HH}^\bullet(M)[+1]^{\mathrm{Lie}}\right) & \xrightarrow{\mathrm{E}_2\ ad.} & \tau_{\geq 1}\left(\mathrm{R\Gamma}(M, \mathrm{Der}^{\mathrm{E}_2}(\mathcal{A}))\right) & \longrightarrow & \tau_{\geq 1}\left(\mathrm{Der}^{\mathrm{E}_2}(\mathbf{HH}^\bullet(M))\right) \\
\downarrow\wr & & \downarrow\wr\, DQ_\Phi & & \\
\tau_{\geq 1}\left(DQ_\Phi(\mathbf{HH}^\bullet(M))[+1]^{\mathrm{Lie}}\right) & \xrightarrow{P_2\ ad.} & \tau_{\geq 1}\left(\mathrm{R\Gamma}(M, \mathrm{Der}^{\mathrm{P}_2}(\mathcal{A}))\right) & \longrightarrow & \tau_{\geq 1}\left(\mathrm{Der}^{\mathrm{P}_2}(DQ_\Phi(\mathbf{HH}^\bullet(M)))\right)
\end{array}$$

and the long horizontal composites are identified with $\tau_{\geq 1}$ of the respective adjoint actions.

Proof.

(i) Applying the first of the two Lemmas below, it suffices to show that $\mathbf{HH}^\bullet_{P_2}(\mathscr{G}) \simeq C^\bullet_{\mathrm{DR}}(M)$. One can verify that the formation of $\mathbf{HH}^\bullet_{P_2}(\mathscr{G})$ has Zariski descent on R, so we are reduced to producing a sheaf map and the affine case. The second Lemma below identifies the sheafy version of $\mathbf{HH}^\bullet_{P_2}(\mathscr{G})$ with $U \mapsto \mathrm{R\Gamma}(U, (\mathrm{Sym}_{\mathscr{G}}\, \mathbb{L}_{\mathscr{G}}[-1], d_{dR}))$. It is well known that the natural map of sheaves of complexes

$$\left(\mathrm{Sym}_{\mathcal{O}_M}\, \mathbb{L}_M[-1], d_{dR}\right) \longrightarrow \left(\mathrm{Sym}_{\mathcal{O}_{\mathscr{G}}}\, \mathbb{L}_{\mathscr{G}}[-1], d_{dR}\right)$$

is a quasi-isomophism of sheaves on M since $\mathscr{G}$ is a nilthickening of $\mathcal{O}_M$.

(c.f., [DTT] for a different way of wording these arguments: There one finds explicit equivalences $\mathscr{G}\text{-mod}^{P_2} \simeq D_{\mathscr{G}}\text{-mod} \simeq D_M\text{-mod}$ carrying $\mathscr{G} \mapsto \mathcal{O}_{\mathscr{G}} \mapsto \mathcal{O}_M$. This identifies $\mathrm{REnd}_{\mathscr{G}\text{-mod}^{P_2}}(\mathscr{G}) = \mathrm{REnd}_{D_M}(\mathcal{O}_M) = C^\bullet_{\mathrm{DR}}(M)$.)

(ii) Follows from (i). To elaborate: one can identify the underlying complex of $\mathrm{Der}^{\mathrm{P}_2}(\mathscr{G})$ with $\mathrm{R\Gamma}\left(M, (\mathrm{Sym}^{\geq 1}_{\mathscr{G}}\, \mathbb{L}_{\mathscr{G}}[-1], d_{int} + d_{dR})\right)[+1]$. Picking a small model for $\mathbb{L}_{\mathscr{G}}$–as $\mathscr{G} \otimes_{\mathcal{O}_M} (\Omega_M + T_M[-1])$ with vanishing differential–the complex of sheaves has nothing above degree 1, and only Ω_M in degree 1 with the outgoing differential being the de Rham differential on M.

(iii) By (ii), all four terms are Lie algebras concentrated in homological degree 1. In particular, they are necessarily abelian and it suffices to check that the corresponding diagram of abelian groups gotten by taking π_1 is commutative. For this, we must compare two maps from $\Gamma(M, \mathcal{O}_M) \to \Gamma(M, (\Omega^1_M)^{\mathrm{closed}})$.

First assume that M is affine. It is enough to show that in this case the natural map

$$\Gamma(M, (\Omega^1_M)^{\mathrm{closed}}) = \pi_1 \mathrm{Der}^{\mathrm{P}_2}(\mathcal{A}) \longrightarrow \pi_1 \mathrm{Der}^{\mathrm{Lie}}(\mathcal{A}[+1])$$

is *injective*, for the composited to the end agree by Theorem 7.1.2.2. But indeed, the derivation of Schouten bracket with a closed form recovers the form as follows: Any such is $\mathcal{O}_M$-linear, so that we may twist by top forms ω_M after which we in particular recover the map $df\wedge\colon \mathcal{O}_M \to \Omega^1_M$.

The general case follows from the affine case: The formation of adjoint actions is compatible with homotopy limits, so that the map we're interested in is gotten by taking global sections of the local maps.

(iv) The right-most square is obviously commutative, as it is just the formation of a homotopy limit of derivations, and similarly for the long horizontal arrows being the adjoint actions. The left-most square commutes because it is gotten from that in (iii) by taking $\tau_{\geq 1}(\mathrm{R}\Gamma(M,-))$. □

Lemma 7.1.3.5. *Suppose $\mathscr{G}$ is a P_n-algebra and let $\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n] = \mathrm{REnd}_{\mathscr{G}\text{-mod}^{P_n}}(\mathscr{G})[n]$ be its (shifted?) operadic cohomology. Then, there is a fiber sequence of Lie algebras $\mathscr{G}[n-1] \overset{ad}{\to} \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G}) \to \mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n]$. De-looping, there is a fiber sequence of Lie algebras $\Omega\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n] \to \mathscr{G}[n-1] \overset{ad}{\to} \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G})$ where $\Omega\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n]$ is quasi-isomorphic to the abelian Lie algebra $\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n-1]$.*

Proof. For $n > 1$, one can deduce this (in a silly roundabout way!) by applying E_n formality and the analogous statements for E_n (see [L6, §6.3.7]). Note also that any dg Lie algebra admitting a delooping is (equivalent to) an abelian Lie algebra.

There is also a more direct approach, using self Koszul duality of the operad P_n up to a shift: Form $C = \mathrm{coFree}^{\mathrm{coP_n}}(\mathscr{G}[+n])$. Then, $\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n]$ identifies–as dg-Lie algebra–with coP_n coderivations of C. Meanwhile, $\mathrm{Der}^{\mathrm{P_n}}(\mathscr{G})$ identifies with the sub dg-Lie algebra of coderivations vanishing on the counit. Evaluation at the counit identifies the quotient complex with $\mathscr{G}[n]$. It follows that there is a fiber sequence, of chain complexes, $\mathscr{G}[n-1] \to \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G}) \overset{ad}{\to} \mathbf{HH}^\bullet_{P_n}(\mathscr{G})[n]$. It remains only to verify that the adjoint action map of dg-Lie algebras is an equivalence onto the fiber. □

Lemma 7.1.3.6. *Suppose $\mathscr{G}$ is a P_n-algebra, and let $\mathscr{G}'$ denote the underlying commutative algebra of $\mathscr{G}$ viewed as a P_n-algebra with the trivial bracket. Then,*

(i) There is a natural equivalence of dg-Lie algebras $\mathbf{HH}^\bullet_{P_n}(\mathscr{G}')[+n] \simeq \left(\widehat{\mathrm{Sym}_{\mathscr{G}'}} T_{\mathscr{G}'}[-n]\right)[+n]$, where the latter is equipped with the Lie bracket coming from that on $T_{\mathscr{G}'}$. The P_n-algebra structure on $\mathscr{G}$ gives rise to a Maurer-Cartan element Π in this ("Poisson bivector"), and there is then an equivalence

$$\mathbf{HH}^\bullet_{P_n}(\mathscr{G}) \simeq \left(\widehat{\mathrm{Sym}_{\mathscr{G}}} T_{\mathscr{G}}[-n], d_{internal} + [\Pi, -]\right)[+n]$$

where $d_{internal}$ is the differential on the underlying complex of $\widehat{\mathrm{Sym}_{\mathscr{G}}} T_{\mathscr{G}}[-n]$ induced by the commutative dga structure on $\mathscr{G}$.

(ii) Contraction against Π induces a map of complexes (even of $\mathbf{HH}^\bullet_{P_n}(\mathscr{G}')[+n]$-modules)

$$i_\Pi(-)\colon \left(\widehat{\mathrm{Sym}_{\mathscr{G}}}\mathbb{L}_{\mathscr{G}}[-1], d_{internal} + d_{deRham}\right) \longrightarrow \left(\widehat{\mathrm{Sym}_{\mathscr{G}}} T_{\mathscr{G}}[-n], d_{internal} + [\Pi, -]\right)$$

(iii) Suppose that the P_n structure is non-degenerate *in the sense that $i_{\Pi_2}\colon \mathbb{L}_{\mathscr{G}}[-1] \to T_{\mathscr{G}}[-n]$ is an equivalence. Then, the map in (ii) is an equivalence.*

Proof. One can identify $\mathbf{HH}^\bullet_{P_n}(\mathscr{G})[+n]$ with $\mathrm{coP_n}$ coderivations of

$$C = \mathrm{coFree}^{\mathrm{coP_2}}(\mathscr{G}[+n]) = \mathrm{coFree}^{\mathrm{coComm}}(\mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[1])[n-1])$$

equipped with a certain differential that may be described as follows. The complex $\mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[1])$ is made into a dg-Lie-bialgebra (with some degree shifts) via a differential d encoding the dg-commutative algebra structure on $\mathscr{G}$ and a bracket gotten by extending $[,] : \mathscr{G}[n-1] \otimes \mathscr{G}[n-1] \to \mathscr{G}[n-1]$ by cobrackets. Then, the total complex is a dg-coP_n-coalgebra via a differential that is the sum of two parts $d = d_m + d_{[,]}$. Here d is the extension by coproducts of d on $\mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[+1])$, while $d_{[,]}$ encodes the Lie bracket on $\mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[+1])$.

The key points are now that the differential always splits into these two pieces, each of which is a differential and which satisfy:

- d_m depends only on the underlying commutative algebra of $\mathscr{G}$, and d_m preserves the grading of $\mathrm{coFree}^{\mathrm{coP_2}}$ gotten by the number of co-commutative terms (i.e., the grading on $\mathrm{coFree}^{\mathrm{coComm}}$).
- $d_{[,]} = 0$ for $\mathscr{G}'$, i.e., if the bracket on $\mathscr{G}$ is trivial.

Let $\mathrm{coDer}_i(C) \subset \mathrm{coDer}(C) \stackrel{\mathrm{def}}{=} \mathrm{coDer}^{\mathrm{coP_2}}(C)$ denote the subspace of those derivations which are allowed to be non-vanishing only on $\mathrm{coFree}^{\mathrm{coComm}}_i(\mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[+1])[n-1]) \subset C$. In general, this gives rise to a decreasing filtration by subcomplexes of $\mathrm{coDer}(C)$ by $F_m\mathrm{coDer}(C) = \prod_{\ell \geq m} \mathrm{coDer}\ell(C)$ – for instance, $F_1 = \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G})[-n]$. Since d_m preserves the $\mathrm{coFree}^{\mathrm{coComm}}_i$ summands, in case the bracket on $\mathscr{G}$ vanishes one gets that the coDer_i make $\mathrm{coDer}(C)$ a graded complex. Finally, there are natural quasi-isomorphisms $\mathrm{Sym}^i_{\mathscr{G}}(T_{\mathscr{G}}[-n])[+n] \simeq \mathrm{coDer}_i(C)$, proving (i).

The proof of (ii) is a computation to verify that the map is compatible with the differential. If the P_n structure is non-degenerate, then the map induces an isomorphism on the associated graded of our favorite decreasing Hausdorff filtration. □

The computation of the above proposition, for $M = \mathbb{A}^n$, was a key input in Tamarkin's first proof of P_2 formality of $DQ_\Phi\mathbf{HH}^\bullet(\mathbb{A}^n)$ in [T1]. It seems plausible that the Calc_2 formality proof in [DTT] can extended to prove compatibility of quantization and adjoint action for $\mathcal{A} = (\mathbf{HH}^\bullet(M), \mathbf{HH}_\bullet(M)) \in \mathrm{E}_2^{calc}$-alg. More precisley, one would ask to show formality for

$$\left(\mathbf{HH}^\bullet(M)[+1]^{\mathrm{Lie}}, DQ_\Phi\left[(\mathbf{HH}^\bullet(M), \mathbf{HH}_\bullet(M))^{\mathrm{ad}}\right]\right)$$

as an algebra over the 3-colored operad of a Lie-algebra $\mathscr{L}$, and a Calc_2-algebra $\mathcal{A}$ on which $\mathscr{L}$ acts by derivations. This would completely circumvent the general Conjecture for our purposes, but we have not carried this out in part because we believe the general Conjecture should hold!

7.1.4 Applications

Finally, we come to the reason we went to all this mess. Applying Theorem 7.1.2.2 and Theorem 7.1.2.1, we have:

Corollary 7.1.4.1. *Suppose $\mathcal{C} \in (\mathbf{dgcat}_k^{\mathrm{idm}})^{B\widehat{\mathbb{G}_a}}$ is a category acted on $B\widehat{\mathbb{G}_a}$. Let $\phi\colon k[+1] \to \mathbf{HH}^\bullet(\mathcal{C})[+1]$ be the Lie map encoding the action. Then,*

– *The Lie $k[+1]$ action on* $\mathbf{HH}^\bullet(\mathcal{C})[+1] \in \mathrm{Lie\text{-}alg}(k\text{-mod})$ *is equivalent to the composite*

$$k[+1] \xrightarrow{\phi} \mathbf{HH}^\bullet(\mathcal{C})[+1] \xrightarrow{\mathrm{ad_{Lie}}} \mathrm{Der}^{\mathrm{Lie}}(\mathbf{HH}^\bullet(\mathcal{C})[+1])$$

of ϕ and the Lie adjoint action.

– *The Lie $k[+1]$ action on* $(\mathbf{HH}^\bullet(\mathcal{C})[+1], \mathbf{HH}_\bullet(\mathcal{C})) \in \mathrm{Lie}[+1]^+_\delta\text{-alg}(k\text{-mod})$ *is equivalent to the composite*

$$k[+1] \xrightarrow{\phi} \mathbf{HH}^\bullet(\mathcal{C})[+1] \xrightarrow{\mathrm{ad}_{\mathrm{Lie}[+1]^+_\delta\text{-alg}}} \mathrm{Der}^{\mathrm{Lie}[+1]^+_\delta\text{-alg}}(\mathbf{HH}^\bullet(\mathcal{C})[+1], \mathbf{HH}_\bullet(\mathcal{C}))$$

of ϕ and the $\mathrm{Lie}[+1]^+_\delta\text{-alg}$ *adjoint action.*

– *The Lie $k[+1]$ action on* $\mathbf{HH}^\bullet(\mathcal{C}) \in \mathrm{E_2\text{-}alg}(k\text{-mod})$ *is equivalent to the composite*

$$k[+1] \xrightarrow{\phi} \mathbf{HH}^\bullet(\mathcal{C})[+1] \xrightarrow{\mathrm{ad_{E2}}} \mathrm{Der}^{\mathrm{E}_2}(\mathbf{HH}^\bullet(\mathcal{C}))$$

of ϕ and the E_2 *adjoint action.*

– *The Lie $k[+1]$ action on* $\mathbf{HH}^\bullet(\mathcal{C})[+1] \in \mathrm{E}_2^{calc}\text{-alg}(k\text{-mod})$ *is equivalent to the composite*

$$k[+1] \xrightarrow{\phi} \mathbf{HH}^\bullet(\mathcal{C})[+1] \xrightarrow{\mathrm{ad_{Lie}}} \mathrm{Der}^{\mathrm{E}_2^{\mathrm{calc}}}(\mathbf{HH}^\bullet(\mathcal{C}), \mathbf{HH}_\bullet(\mathcal{C})])$$

of ϕ and the E_2^{calc} *adjoint action.*

Applying the Lie- and $\mathrm{Lie}[+1]^+_\delta$-alg- formalities on M, we get:

Corollary 7.1.4.2. *Suppose (M, f) is an LG pair, $\mathcal{C} = \mathrm{DCoh}(M)$ acted on by $B\widehat{\mathbb{G}_a}$ according to f. Then,*

– *There is an equivalence of* $\mathrm{Lie}[+1]^+_\delta\text{-alg}(k\text{-mod})$ *acted on by $k[+1]$:*

$$\begin{aligned}&\phi^* \left(\mathbf{HH}^\bullet(\mathcal{C})[+1], \mathbf{HH}_\bullet(\mathcal{C}), \delta = \textit{Connes } B\right)^{\mathrm{ad}_{\mathrm{Lie}[+1]^+_\delta}} \\ &\quad\simeq \phi^* \left(\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1])[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1]), \delta = d_{dR}\right)^{\mathrm{ad}_{\mathrm{Lie}[+1]^+_\delta}}\end{aligned}$$

where on the right we have the usual operations of polyvector fields, so that for instance the B-operator of the $k[+1]$-action is the Lie action by f.

– *There is an equivalence in* $\mathrm{Lie}[+1]^+_\delta\text{-alg}(k(\!(\beta)\!))$

$$\begin{aligned}&(\mathbf{HH}^\bullet(\mathcal{C})[+1], \mathbf{HH}_\bullet(\mathcal{C}), B)^{B\widehat{\mathbb{G}_a}} \\ &\simeq (\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1](\!(\beta)\!), \beta \cdot i_{df})[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1](\!(\beta)\!), -\beta df \wedge), d_{dR})\end{aligned}$$

and an equivalence in $\mathrm{Lie}[+1]^+\text{-alg}(k(\!(\beta)\!))$

$$\begin{aligned}&\left(\mathbf{HH}^\bullet(\mathcal{C})[+1]^{B\widehat{\mathbb{G}_a}}, \mathbf{HH}_\bullet(\mathcal{C})^{B\widehat{\mathbb{G}_a}\times \mathrm{SO}(2)}\right) \\ &\simeq (\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1](\!(\beta)\!), \beta \cdot i_{df})[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1](\!(\beta)\!)(\!(u)\!), u \cdot d_{dR} - \beta df \wedge), d_{dR})\,.\end{aligned}$$

Applying the E_2^{calc}-formality on M along with Prop. 7.1.3.4, we get:

Corollary 7.1.4.3. *Suppose* (M, f) *is an LG pair,* $\mathcal{C} = \mathrm{DCoh}(M)$ *acted on by* $B\widehat{\mathbb{G}_a}$ *according to* f*. Then,*

- *There is an equivalence of* Calc_2-alg(k-mod) *acted on by* $k[+1]$*:*

$$\begin{aligned}&\phi^* \,\mathrm{QC}_\Phi \left(\mathbf{HH}^\bullet(\mathcal{C}), \mathbf{HH}_\bullet(\mathcal{C}), \delta = \textit{Connes } B\right)^{\mathrm{ad}_{\mathrm{E}_2^{calc}}}\\ &\quad\simeq \phi^* \left(\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1])[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1]), \delta = d_{dR}\right)^{\mathrm{ad}_{\mathrm{Calc}_2}}\end{aligned}$$

where on the right we have the usual operations of polyvector fields, so that for instance the B*-operator of the* $k[+1]$*-action is the Lie action by* f*.*

- *There is an equivalence in* Calc_2-alg($k(\!(\beta)\!)$-mod)

$$\begin{aligned}&(\mathbf{HH}^\bullet(\mathcal{C})[+1], \mathbf{HH}_\bullet(\mathcal{C}), B)^{B\widehat{\mathbb{G}_a}}\\ &\simeq (\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1](\!(\beta)\!), \beta \cdot i_{df})[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1](\!(\beta)\!), -\beta df\wedge), d_{dR})\end{aligned}$$

and an equivalence in $\mathscr{P}_2^+$-alg($k(\!(\beta)\!)$)

$$\begin{aligned}&\left(\mathbf{HH}^\bullet(\mathcal{C})[+1]^{B\widehat{\mathbb{G}_a}}, \mathbf{HH}_\bullet(\mathcal{C})^{B\widehat{\mathbb{G}_a}\times \mathrm{SO}(2)}\right)\\ &\simeq (\mathrm{R\Gamma}(\mathrm{Sym}_M T_M[-1](\!(\beta)\!), \beta \cdot i_{df})[+1], \mathrm{R\Gamma}(\mathrm{Sym}_M \Omega_M[+1](\!(\beta)\!)(\!(u)\!), u \cdot d_{dR} - \beta df\wedge), d_{dR})\,.\end{aligned}$$

7.2 E_n-algebras, P_n-algebras, and Calculus-variants

7.2.1 Generator-and-relation operads

We first recall the definitions of, and fix out notation for, several operads defined by generators and relations:

Definition 7.2.1.1. For each $n \in \mathbb{Z}$, there is the operad $\mathrm{Lie}[n]$ of (shifted) Lie algebras is generated by a single binary operation $[-,-]$ of degree n satisfying (graded skew-symmetry) $[x, y] = -[y, x]$ and (Jacobi identity) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$. An algebra for $\mathrm{Lie}[n]$ is a complex L such that $L[+n]$ is a dg-Lie algebra. Note that the Jacobi identity is precisely the statement that $[x, -]$ is a derivation of the bracket.

Definition 7.2.1.2. There is the 2-colored operad $\mathrm{Lie}[+1]^+_\delta$ whose algebras are pairs (L, M) such that $L[+1]$ is a dg-Lie algebra, M is a dg-Lie module over $\oplus M$, and M is equipped with a unary operation δ of degree 1 satisfying $\delta^2 = 0$ and such that $\delta([a, -]) = [a, \delta(-)]$. There is an obvious map of operads $\mathrm{Lie}[+1] \to \mathrm{Lie}[+1]^+_\delta$ by forgetting the M color.

Definition 7.2.1.3. For each $n \in \mathbb{Z}$, there is the operad P_n of n-Poisson algebras is generated by two binary 2 operations $- \cdot -$ of degree 0 and $[-,-]$ of degree $(n-1)$. The operation $- \cdot -$ satisfies the relations of a commutative product, the operation $[-,-]$ satisfies the relations of a Lie bracket, and there is a further compatibility: $[x, -]$ is a (graded) derivation for the commutative product. There is an obvious morphism of operads $\mathrm{Lie}[n-1] \to P_n$ by forgetting the commutative product.

Definition 7.2.1.4. There is the 2-colored operad Calc_2 whose algebras are pairs $(\mathscr{G}, M)$ such that $\mathscr{G}$ is a P_2-algebra, M is a $\mathscr{G}$-module, δ is a unary operation of degree 1 satisfying

$\delta^2 = 0$ and $[\delta, a] = [a, -]$. There is a map of operads $\mathrm{Lie}[1]^+_\delta \to \mathrm{Calc}_2$ by taking the Lie operation and Lie module structure, and δ to δ.[1]

Remark 7.2.1.5. Each of the above operads is *graded*. In all cases, the commutative operations have degree 0,the Lie operations have degree $n-1$, and δ has degree 1. If $n \geq 1$, they are positively graded.

7.2.2 Topological operads

Now a few operads from topology:

Definition 7.2.2.1. For each $n \geq 1$, there is the operad E_n of chains on the configuration spaces of little disks. For all $n \geq 1$, this is a filtered operad with $\mathrm{gr} F_\bullet \mathrm{E}_n = P_n$. For $n \geq 2$, the filtration is the Postnikov filtration $F_{-i}V = \tau_{\geq i} V$. For $n = 1$, it is the usual filtration on the associative operad having associated graded the Poisson operad.

Definition 7.2.2.2. For each $n \geq 1$, there is the 2-colored operad E_n^{calc} whose algebras are pairs (A, M) where A is a E_n-algebra, and M is a fE_n-module over A (i.e., a complex M with $\mathrm{SO}(n)$-action together with $\mathrm{SO}(n)$-equivariant maps $\mathrm{E}_n(m) \otimes \mathcal{A}^{\otimes m-1} \otimes M \to M$ giving M an $\mathcal{A}$-module structure compatibly with the $\mathrm{SO}(n)$-action on $\mathrm{E}_n(m)$). For all $n \geq 1$, E_n^{calc} is a filtered operad with $\mathrm{gr} F_\bullet \mathrm{E}_n^{calc} = \mathrm{Calc}_n$. (So, for instance, the B_i and δ correspond to the standard generators for $H_*(\mathrm{SO}(n))$.) For $n = 2$, there are several models for this in [KS, §11].

Remark 7.2.2.3. As written, the above are various operads in (literal) chain complexes. We will often regard them as objects in an $(\infty, 1)$-category of ∞-operads: In that context, cofibrant replacement is implicit and we will not give explicit names to the cofibrant replacement operads. For instance, the notation E_n-alg, E_n-alg(k-mod), Lie-alg, Calc_2-alg(R-mod), etc. will almost always be intended in this sense. That said, when we write "a dg Lie algebra" we may sometimes mean a literal dg Lie algebra (with a well-defined underlying graded abelian group, etc.).

7.2.2.4. There are natural maps $\mathrm{Lie}[n-1] \to \mathrm{E}_n$ and $\mathrm{Lie}[1]^+_\delta \to \mathrm{E}_2^{calc}$. These can be seen in two ways: Either as the inclusion of the first piece of the filtration mentioned above (which guaranteed uniqueness up to $\mathrm{Aut}(\mathrm{Lie}) = \mathbb{Q}^\times$). Or, more conceptually: as the map Koszul dual to the $\mathrm{E}_n \to \mathrm{Comm}$ under the self-Koszul-dualities $\mathrm{KD}(\mathrm{Comm}) = \mathrm{Lie}[-1]$ and $\mathrm{KD}(\mathrm{E}_n) \simeq \mathrm{E}_n[-n]$.

7.3 Units and adjoint actions for loop spaces, E_k-algebras, etc.

7.3.1 Reminder: categorical delooping machine

Suppose $\mathcal{A}$ is an E_k-algebra in a presentable E_k-monoidal $(\infty, 1)$-category $\mathcal{C}^\otimes$. It would be convenient to think of "$B^k\mathcal{A}$" as a $\mathcal{C}$-*enriched* (∞, k)-*category*, with a single i-morphism for

[1]There are also operads Calc_n for other n. In all cases there is a P_n-algebra, and a $\mathscr{G}$-module. If n is even, then there is a unary operator δ of degree $(n-1)$ and unary operators B_i, $i = 1, \ldots, (n-2)/2$ of degree $4i-1$. If n is odd, there are unary operators B_i, $i = 1, \ldots, (n-1)/2$ of degree $4i-1$. All the unary operations square to zero; the B_i are maps of $\mathscr{G}$-modules; and if δ exists, it satisfies the same relations as in Calc_2.↑

$0 \le i < k$, and with $\mathcal{A}$ worth of k-morphisms. The theory of such enriched higher categories doesn't seem to be written down, but we can approximate what we need from it by repeated formation of module categories:

- mod-$\mathcal{A} \overset{\text{def}}{=} \mathbf{RMod}_{\mathcal{A}}(\mathcal{C})$–the ∞-category of right $\mathcal{A}$-modules in $\mathcal{C}$–is an E_{k-1}-algebra in the E_{k-1}-monoidal $(\infty, 1)$-category mod-$\mathcal{C}^{\otimes} \overset{\text{def}}{=} \mathbf{RMod}_{\mathcal{C}}(\mathrm{Pr}^L)$.
- If $k \ge 2$, we thus iterate this and define mod^2-$\mathcal{A} \overset{\text{def}}{=} \mathbf{RMod}_{\text{mod-}\mathcal{A}}(\text{mod-}\mathcal{C})$. This is a version of the theory of $\mathcal{A}$-linear categories.
- We can continue iterating this procedure, and for each $1 \le i \le k$ we can define

$$\mathrm{mod}^i\text{-}\mathcal{C} \overset{\text{def}}{=} \mathbf{RMod}_{\mathrm{mod}^{i-1}\text{-}\mathcal{C}}(\mathrm{Pr}^L) \in E_{k-i}\text{-alg}(\mathrm{Pr}^L)$$

$$\mathrm{mod}^i\text{-}\mathcal{A} \overset{\text{def}}{=} \mathbf{RMod}_{\mathrm{mod}^{k-1}\text{-}\mathcal{A}}(\mathrm{mod}^{k-1}\text{-}\mathcal{C}) \in E_{k-i}\text{-alg}(\mathrm{mod}^i\text{-}\mathcal{C})$$

  with the base case

$$\mathrm{mod}^0\text{-}\mathcal{C} \overset{\text{def}}{=} \mathcal{C} \in E_k\text{-alg}(\mathrm{Pr}^L) \qquad \text{and} \qquad \mathrm{mod}^0\text{-}\mathcal{A} \overset{\text{def}}{=} \mathcal{A} \in E_k\text{-alg}(\mathcal{C}).$$

- For each $i < k$, note that mod^i-$\mathcal{A}$ is the monoidal unit in the E_{k-i-1}-monoidal mod^{i+1}-$\mathcal{A}$. Taking its mod^i-$\mathcal{C}$-enriched endomorphisms, one obtains $\mathrm{REnd}^{\otimes}_{\mathrm{mod}^{i+1}\text{-}\mathcal{A}}(\mathrm{mod}^i\text{-}\mathcal{A}) = \mathrm{mod}^i\text{-}\mathcal{A} \in E_{k-i}\text{-alg}(\mathrm{mod}^i\text{-}\mathcal{C})$.

Remark 7.3.1.1. Disclaimer: In order to maintain presentability we need to worry about set-theoretic issues that have been ignored above–literally, mod-$\mathcal{C}$ is too big to be presentable. This can be dealt with, and we refer the reader to [L6, §6.3.7].

Remark 7.3.1.2. The above construction determines a sequence of fully-faithful embeddings

$$E_k\text{-alg}(\mathcal{C}) \overset{\mathbf{RMod}}{\hookrightarrow} E_{k-1}\text{-alg}(\text{mod-}\mathcal{C}) \overset{\mathbf{RMod}}{\hookrightarrow} E_{k-2}\text{-alg}(\mathrm{mod}^2\text{-}\mathcal{C}) \overset{\mathbf{RMod}}{\hookrightarrow} \cdots \overset{\mathbf{RMod}}{\hookrightarrow} E_0\text{-alg}(\mathrm{mod}^k\text{-}\mathcal{C}).$$

7.3.2 Delooped space of units and adjoint action, for $k = 1$

If $\mathcal{A}$ is an ordinary algebra, then the group $\mathcal{A}^{\times}$ acts by algebra automorphisms on $\mathcal{A}$ via the conjugation action: $g \cdot a = gag^{-1}$. Let's make sense of this for E_1-algebras:

7.3.2.1. Suppose $\mathcal{A}$ is an E_1-algebra in $\mathcal{C}$. It has a "(multiplicative) underlying E_1-space," $|\mathcal{A}| \overset{\text{def}}{=} \mathrm{Map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathcal{A})$ – the E_1-structure comes from tensoring maps and using the multiplication on $\mathcal{A}$. One can define the grouplike E_1-space $\mathcal{A}^{\times}$ to be the union of those components of $|\mathcal{A}|$ which are invertible in the monoid $\pi_0(|\mathcal{A}|)$.

One can think of the delooping of $\mathcal{A}^{\times}$ in terms of moduli of objects in mod-$\mathcal{A}$:

- Note that $\mathcal{A} \simeq \mathrm{RHom}^{\otimes_{\mathcal{C}}}_{\text{mod-}\mathcal{A}}(\mathcal{A}, \mathcal{A})$. So, $|\mathcal{A}| \simeq \mathrm{Map}_{\text{mod-}\mathcal{A}}(\mathcal{A}, \mathcal{A})$ and $\mathcal{A}^{\times} \simeq \mathrm{Aut}_{\text{mod-}\mathcal{A}}(\mathcal{A})$.
- Form the *space of objects*, $(\text{mod-}\mathcal{A})^{\sim} \in$ **Spaces**: This is the maximal sub-∞-groupoid of mod-$\mathcal{A}$, gotten by discarding all non-invertible 1-morphisms of mod-$\mathcal{A}$. It is a pointed space, by virtue of $\mathcal{A}$.
- By (i), we see that $\Omega(\text{mod-}\mathcal{A})^{\sim} = \mathcal{A}^{\times}$ and that $B\mathcal{A}^{\times}$ is just the connected component of $(\text{mod-}\mathcal{A})^{\sim}$ containing $\mathcal{A}$ – more colloquially, its the moduli space of twists of $\mathcal{A} \in$ mod-$\mathcal{A}$.

7.3.2.2. We now wish to define a map of grouplike E_1-spaces $\mathcal{A}^\times \to \mathrm{Aut}_{E_1}(\mathcal{A})$. We may do this by delooping, and constructing a map of pointed spaces $B\mathcal{A}^\times \to B\,\mathrm{Aut}_{E_1}(\mathcal{A})$: So to a family of twists of $\mathcal{A} \in$ mod-$\mathcal{A}$, we must associate a family of twists of $\mathcal{A} \in E_1$-alg$(\mathcal{C})$.

We do this using functoriality, under isomorphisms, of $\mathcal{C}$-enriched endomorphisms

$$\mathrm{End}^{\otimes_{\mathcal{C}}}_{\text{mod-}\mathcal{A}}(-) \colon (\text{mod-}\mathcal{A})^{\sim} \longrightarrow (E_1\text{-alg}(\mathcal{C}))^{\sim}$$

and restricting to the connected components containing $\mathcal{A} \in$ mod-$\mathcal{A}$ and $\mathcal{A} \in E_1$-alg$(\mathcal{C})$.

A word on building this functoriality: One construction is as the composite

$$(\text{mod-}\mathcal{A})^{\sim} \xrightarrow{F} E_0\text{-alg}(\text{mod-}\mathcal{C}) \xrightarrow{G} E_1\text{-alg}(\mathcal{C})$$

where $F(M) = M \in$ mod-$\mathcal{A}$, and G is the right-adjoint to $\mathbf{RMod}\colon E_1\text{-alg}(\mathcal{C}) \hookrightarrow E_0\text{-alg}(\text{mod-}\mathcal{C})$.

7.3.3 Higher deloopings and adjoint actions, $n > 1$

7.3.3.1. If $\mathcal{A}$ is an E_n-algebra in $\mathcal{C}$, then $|\mathcal{A}|$ (resp., $\mathcal{A}^\times$) is in fact an E_n-space (resp., grouplike) in the same manner as above. One can construct the n-fold delooping of $\mathcal{A}^\times$ in terms of moduli of objects in mod^n-$\mathcal{A}$, as follows:

Note that, $(\text{mod}^n\text{-}\mathcal{A})^{\sim}$ is a pointed space via $(\text{mod-}n-1\mathcal{A})$. For each $1 \leq i \leq n$, one has an inclusion of spaces

$$\Omega^i(\text{mod}^n\text{-}\mathcal{A})^{\sim} \subset (\text{mod}^{n-i}\text{-}\mathcal{A})^{\sim}$$

that can be described as follows: for $i \geq 1$, $(\text{mod}^{n-i}\text{-}\mathcal{A})^{\sim}$ is an E_i-space and $\Omega^i(\text{mod}^n\text{-}\mathcal{A})^{\sim}$ is the inclusion of those components which are units in π_0. In particular, we find that

$$\Omega^n(\text{mod}^n\text{-}\mathcal{A})^{\sim} = \mathcal{A}^\times \subset (\text{mod}^0\text{-}\mathcal{A})^{\sim} = |\mathcal{A}|$$

is the space of units. So, we obtain a map

$$B^n\mathcal{A}^\times = B^n\Omega^n(\text{mod}^n\text{-}\mathcal{A})^{\sim} \longrightarrow (\text{mod}^n\text{-}\mathcal{A})^{\sim}.$$

7.3.3.2. The adjoint action will now be a group map $B^{n-1}\mathcal{A}^\times \to \mathrm{Aut}_{E_n}(\mathcal{A})$, or rather its delooping: A map of pointed spaces

$$B^n\mathcal{A}^\times \to B\,\mathrm{Aut}_{E_n}(\mathcal{A}).$$

So to a family of twists of mod^{n-1}-$\mathcal{A} \in$ mod^n-$\mathcal{A}$, we must associate a family of twists of $\mathcal{A} \in E_n$-alg$(\mathcal{C})$.

We do this again by using functoriality of a $\mathcal{C}$-enriched "higher endomorphism" construction and passing to $(n-1)$-connected covers over the basepoints. As before, one construction of this "higher endomorphism" construction is as the composite

$$\mathrm{End}^n_{/\mathcal{A}} \colon [\text{mod}^n\text{-}\mathcal{A}]^{\sim} \xrightarrow{F} E_0\text{-alg}(\text{mod}^n\text{-}\mathcal{C}) \xrightarrow{G} \mathrm{E}_n\text{-alg}(\mathcal{C})$$

where $F(M) = M \in$ mod^n-$\mathcal{A}$, and G is the right-adjoint to $\mathbf{RMod}\colon E_1\text{-alg}(\mathcal{C}) \hookrightarrow E_0\text{-alg}(\text{mod-}\mathcal{C})$.

7.3.3.3. This construction is also functorial in $\mathcal{A}$ in the following sense:

– Let $[\text{mod}^n\text{-}-]^{\sim}$ denote the $(\infty, 1)$-category of pairs $(\mathcal{A}, \mathcal{C})$ of $\mathcal{A} \in \mathrm{E}_n$-alg and $\mathcal{C} \in$ mod^n-$\mathcal{A}$. A morphism of pairs $(\mathcal{A}, \mathcal{C}) \to (\mathcal{A}', \mathcal{C}')$ consists of a map $\mathcal{A} \to \mathcal{A}'$ of E_n-

algebras, and an $\mathcal{A}$-linear map $\mathcal{C} \to \mathcal{C}'$ over it *which is required to induce an equivalence* $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}' \to \mathcal{C}'$.

– Then, there is a functor $[\mathrm{mod}^n\text{-}-]^{\sim} \to \mathrm{E}_n\text{-alg}$ given heuristically by $(\mathcal{A}, \mathcal{C}) \mapsto \mathrm{End}^n_{/\mathcal{A}}(\mathcal{C})$ or more precisely by the composite $(\mathcal{A}, \mathcal{C}) \mapsto G\,(\mathcal{C} \in \mathrm{mod}^n\text{-}\mathcal{A})$ with G as above.

Remark 7.3.3.4. Suppose that $\mathcal{C}$ is chain complexes and that $n = 2$. In this case, $\mathcal{A}$ is an ordinary E_2-algebra and $\mathrm{mod}^2\text{-}\mathcal{A}$ is the theory of $\mathcal{A}$-linear dg-categories (recall we're being sloppy about size and colimit preservation). In this case, the higher endomorphism construction is something familiar: The $\mathcal{A}$-linear Hochschild cochains

$$\mathrm{End}^2_{/\mathcal{A}} = \mathbf{HH}^{\bullet}_{/\mathcal{A}} \colon (\mathbf{dgcat}_{/\mathcal{A}})^{\sim} \longrightarrow \mathrm{E}_2\text{-alg}$$

7.3.4 Units and augmentations

7.3.4.1. Thus far, we have only considered the case where $\mathcal{A}$ was a *unital* E_n-algebra. However, this restriction is not material: Suppose $\mathfrak{m}_{\mathcal{A}}$ is a nonunital E_n-algebra, and let $\mathcal{A} = \mathbb{1} \oplus \mathfrak{m}_{\mathcal{A}}$ denote the augmented E_n-algebra generated by it. We will now describe an augmented version of the adjoint action:

7.3.4.2. Suppose $\mathcal{A} \to \mathbb{1}_{\mathcal{C}}$ is an augmented E_n-algebra in $\mathcal{C}^{\otimes}$. Then, tensoring along the augmentation we obtain a diagram

$$\begin{array}{ccc} (\mathrm{mod}^n\text{-}\mathcal{A})^{\sim} & \xrightarrow{\mathrm{End}^n} & \mathrm{E}_n\text{-alg}(\mathcal{C}) \\ {\scriptstyle -\otimes_{\mathrm{mod}^{n-1}\text{-}\mathcal{A}}\mathrm{mod}^{n-1}\text{-}\mathbb{1}_{\mathcal{C}}}\downarrow & \swArrow & \downarrow{\scriptstyle -\otimes_{\mathcal{A}}\mathbb{1}_{\mathcal{C}}} \\ (\mathrm{mod}^n\text{-}\mathbb{1}_{\mathcal{C}})^{\sim} & \xrightarrow[\mathrm{End}^n]{} & \mathrm{E}_n\text{-alg}(\mathcal{C}) \end{array}$$

or in other words, a natural transformation of functors $\mathrm{End}^n_{/\mathcal{A}} \to \mathbb{1}_{\mathcal{C}}$ making $\mathrm{End}^n_{/\mathcal{A}}$ land in augmented E_n-algebras. This allows us to define a reduced version

$$\overline{\mathrm{End}}^n_{/\mathcal{A}} \stackrel{\mathrm{def}}{=} \mathrm{fib}\{\mathrm{End}^n_{/\mathcal{A}} \to \mathbb{1}_{\mathcal{C}}\} \colon (\mathrm{mod}^n\text{-}\mathcal{A})^{\sim} \longrightarrow \mathrm{E}^{nu}_n\text{-alg}(\mathcal{C})$$

such that $\overline{\mathrm{End}}^n_{/\mathcal{A}}(\mathcal{A}) = \mathfrak{m}_{\mathcal{A}}$. Consequently, we obtain a *non-unital (or augmented) adjoint action*

$$\overline{\mathrm{End}}^n_{/\mathcal{A}} \colon B^n(k \oplus \mathcal{A})^{\times} \longrightarrow B\,\mathrm{Aut}_{\mathrm{E}^{nu}_n}(\mathfrak{m}_{\mathcal{A}}) \subset \mathrm{E}^{nu}_n\text{-alg}(\mathcal{C})^{\sim}.$$

7.3.4.3. There is one further property of the non-unital adjoint action that we'd like to record. The inclusion $\mathbb{1} \hookrightarrow \mathbb{1} \oplus \mathcal{A}$ induces $\mathbb{1}^{\times} \hookrightarrow (\mathbb{1} \oplus \mathcal{A})^{\otimes}$. The restriction of the adjoint action to $\mathbb{1}$ is trivialized. This follows from the discussion in [L6, 6.3.7], since $\mathbb{1} \to \mathcal{A}$ factors canonically through $\mathfrak{Z}_{\mathrm{E}_n}(\mathrm{id}_{\mathcal{A}})$.

7.3.5 Variation with a module or Calculus-type structure

7.3.5.1. Suppose that $\mathcal{A}$ is an E_n-algebra. We will have reason to consider the pair of (not adjoint) functors

$$\mathcal{A}\text{-mod}^{\mathrm{E}_n} \underset{I}{\overset{\mathcal{A}\oplus -}{\rightleftarrows}} \mathrm{E}_n\text{-alg}_{/\mathcal{A}/}$$

where I denotes the augmentation ideal.

There is also an evident where the algebra is allowed to vary:

– Let E_n-mod denote the ∞-category of pairs $(\mathcal{A}, M)$ of an E_n-algebra $\mathcal{A}$ and an E_n-module M over $\mathcal{A}$.

– Let $\mathrm{E}_n\text{-alg}_{//}$ denote the ∞-category of pairs $(\mathcal{A}, \mathcal{A}')$ of an E_n-algebra $\mathcal{A}$ and an E_n-algebra $\mathcal{A}'$ over and under $\mathcal{A}$.

– Then, there are functors $- \oplus -\colon \mathrm{E}_n\text{-mod} \to \mathrm{E}_n\text{-alg}_{//}\colon I$ over the forgetful functor to E_n-alg.

7.3.5.2. Suppose that $(\mathcal{A}, M)$ is a pair of an E_n-algebra and an E_n-module M over $\mathcal{A}$. Then, $\mathcal{A} \oplus M \in \mathrm{E}_n\text{-alg}_{/\mathcal{A}/}$ is an E_n-algebra over and under $\mathcal{A}$. The above gives an adjoint action of $B^{n-1}(\mathcal{A} \oplus M)^\times$ on $(\mathcal{A} \oplus M)$. Furthermore, the restriction along $B^{n-1}\mathcal{A}^\times \to B^{n-1}(\mathcal{A} \oplus M)^\times$ preserves the augmentation so that we obtain

$$B^{n-1}\mathcal{A}^\times \longrightarrow \mathrm{Aut}_{\mathrm{E}_n\text{-alg}_{//}}(\mathcal{A} \oplus M) \xrightarrow{I} \mathrm{Aut}_{\mathrm{E}_n\text{-mod}}(\mathcal{A}, M)$$

This is the E_n^+ *adjoint action* for $(\mathcal{A}, M)$.

7.3.5.3. Suppose now that $(\mathcal{A}, M) \in \mathrm{E}_n^{calc}$-alg. This provides $\mathcal{A} \oplus M$ with equivariance isomorphisms $\mathcal{A} \oplus M \xrightarrow{\sim} g^{-1}(g(\mathcal{A}) \oplus M)$ for $g \in \mathrm{SO}(n-1)$. Applying the above construction to this diagram, one can construct the E_n^{calc} *adjoint action* for $(\mathcal{A}, M)$

$$B^{n-1}\mathcal{A}^\times \longrightarrow \mathrm{Aut}_{\mathrm{E}_n^{calc}\text{-mod}}(\mathcal{A}, M)$$

7.3.6 Universality for $n = 2$

We have the discrete analogue of Theorem 7.1.2.1.

Theorem 7.3.6.1. *Suppose* $\mathcal{C} \in \mathbf{dgcat}^{\mathrm{idm}}$. *Let* $\mathcal{A} = \mathbf{HH}^\bullet(\mathcal{C}) \in \mathrm{E}_2\text{-alg}(k\text{-mod})$. *Then,* $B(\mathbf{HH}^\bullet(\mathcal{C})^\times)$ *acts on* $\mathcal{C}$ *and consequently on* $\mathcal{A} \in \mathrm{E}_2$-alg. *This action may be naturally identified, up to homotopy, with the* E_2 adjoint action *described above. In particular, it depends only on the* E_2*-algebra* $\mathcal{A}$ *and not on the category* $\mathcal{C}$.

Proof. The identity map $\mathcal{A} \to \mathbf{HH}^\bullet(\mathcal{C})$ lifts $\mathcal{C}$ to an object of $\mathbf{dgcat}^{\mathrm{idm}}_{\mathcal{A}}$. Consequently, we have a diagram

$$\begin{array}{ccccc}
B^2(\mathbf{HH}^\bullet_{/\mathcal{A}}(\operatorname{Perf}\mathcal{A})^\times) & \xrightarrow{\operatorname{Perf}\mathcal{A}} & (\text{mod-}(\operatorname{Perf} A))^\sim & \xrightarrow{\mathbf{HH}^\bullet_{/\mathcal{A}}} & \mathrm{E}_2\text{-alg}(k\text{-mod}) \\
\downarrow\wr & & \downarrow{\scriptstyle -\otimes_{\operatorname{Perf} A}\mathcal{C}} & & \| \\
B^2(\mathbf{HH}^\bullet_{/k}(\mathcal{C})^\times) & \xrightarrow[\mathcal{C}]{} & (\mathbf{dgcat}^{\mathrm{idm}})^\sim & \xrightarrow[\mathbf{HH}^\bullet_{/k}]{} & \mathrm{E}_2\text{-alg}(k\text{-mod})
\end{array}$$

i.e., an equivalence $\operatorname{Perf}\mathcal{A} \otimes_{\operatorname{Perf}\mathcal{A}} \mathcal{C} \to \mathcal{C}$ natural in $\mathcal{A}$-linear automorphisms of $\mathrm{id}_{\operatorname{Perf}\mathcal{A}}$; and, a natural transformation $\mathbf{HH}^\bullet_{/\mathcal{A}}(-) \to \mathbf{HH}^\bullet(- \otimes_{\mathcal{A}} \mathcal{C})$. This second natural transformation is an equivalence on objects equivalent to $\operatorname{Perf}\mathcal{A}$. Consequently, we have produced an equivalence of functors between the two horizontal composite arrows: The one on the top is the adjoint action for $\mathbf{HH}^\bullet(\mathcal{C})$ and the one on the bottom is the one induced by the action of $B(\mathbf{HH}^\bullet(\mathcal{C})^\times)$ on $\mathcal{C}$. ☐

The same proof also establishes, up to set-theoretic issues that we're neglecting, the higher n analog:

Theorem 7.3.6.2. *Fix $n \geq 1$, and suppose $\mathcal{C} \in \mathrm{mod}^n\text{-}k$. Let $\mathcal{A} = \mathrm{End}^n(\mathcal{C}) \in \mathrm{E}_n\text{-alg}(k\text{-mod})$. Then, $B^{n-1}(\mathcal{A}^\times)$ acts on $\mathcal{C}$ and consequently on $\mathcal{A} \in \mathrm{E}_n\text{-alg}$. This action may be naturally identified, up to homotopy, with the* E_n adjoint action *described above. In particular, it depends only on the E_n-algebra $\mathcal{A}$ and not on $\mathcal{C}$.*

Proof. Look at

$$\begin{array}{ccccc}
B^n(\mathrm{End}^n_{/\mathcal{A}}(\mathrm{mod}^{n-1}\text{-}\mathcal{A})^\times) & \xrightarrow{\mathrm{mod}^{n-1}\text{-}\mathcal{A}} & (\mathrm{mod}^n\text{-}\mathcal{A})^\sim & \xrightarrow{\mathrm{End}^n_{/\mathcal{A}}} & \mathrm{E}_n\text{-alg}(k\text{-mod}) \\
\big\downarrow{\scriptstyle\sim} & \swarrow\!\!\swarrow{\scriptstyle\sim} & \big\downarrow{\scriptstyle -\otimes_{\mathrm{mod}^{n-1}\text{-}\mathcal{A}}\mathcal{C}} & \nearrow\!\!\nearrow & \big\| \\
B^n(\mathrm{End}^n_{/k}(\mathcal{C})^\times) & \xrightarrow[\mathcal{C}]{} & (\mathrm{mod}^n\text{-}k)^\sim & \xrightarrow[\mathrm{End}^n_{/k}]{} & \mathrm{E}_n\text{-alg}(k\text{-mod})
\end{array}$$

□

7.4 Infinitesimal adjoint actions for Lie-, E_n-, P_n-, etc. algebras

7.4.1 Some (pre-)formal moduli problems

We will only really need the following for $n = 2$, but will state it more generally than we use or prove:

Proposition 7.4.1.1. *Suppose $\mathcal{A}$ is an E_n-algebra as above, and consider the E_n pre-fmp $\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}^{pre}$ given on $A \in$ **DArt** by*

$$\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}^{pre}(A) = \left\{ M \in \mathrm{mod}^n\text{-}(\mathcal{A} \otimes A), \phi\colon M \otimes_A n \simeq \mathrm{mod}^{n-1}\text{-}\mathcal{A} \right\}$$

Then,

(i) $\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}^{pre}$ is a n-proximate-fmp in the terminology of [L5], i.e., Ω^n of it is a fmp;

(ii) There is a natural equivalence of non-unital E_n-algebras

$$T^{\mathrm{E}_n}[-n]_{\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}} \xrightarrow{\sim} \mathcal{A}$$

(iii) Letting $\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}$ denote also the restriction of the above to a commutative pre-fmp, this equips $\mathcal{A}[n-1]$ with the structure of Lie algebra vie the identification

$$T^{\mathrm{Lie}}[-1]_{\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}} \xrightarrow{\sim} \mathcal{A}[n-1]$$

Proof. The first point follows by an argument analogous to Prop. 5.3.3.4(i): Identify $\Omega^n\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}^{pre}$ with $\widehat{\mathrm{mod}^0\text{-}\mathcal{A}}^{pre}$, and then show that the latter is a formal moduli problem, with $\mathrm{End}^n_{\mathcal{A}}(\mathrm{mod}^{n-1}\text{-}\mathcal{A}) \simeq \mathcal{A}$ taking the role of $\mathbf{HH}^\bullet(\mathcal{C}_0)$. The third point is a definition, rather than a claim. For the second point, in light of (i) it suffices to provide a morphism from $\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}$ to the fmp corresponding to $\mathcal{A}$

$$A \mapsto \mathrm{Map}(\mathrm{KD}^{\mathrm{E}_n}(A), \mathcal{A})$$

At least for $n = 2$, which is all we will need here, it can be done in a manner analogous to that of [L5, 5.3.18]. $\square$

Definition 7.4.1.2. Consider the commutative pre-fmp $\mathrm{E}_n\text{-alg}_{\widehat{\mathcal{A}}}$ given on $A \in \mathbf{DArt}$ by

$$\mathrm{E}_n\text{-alg}_{\widehat{\mathcal{A}}}^{pre} = \left\{ \widetilde{\mathcal{A}} \in \mathrm{E}_n\text{-alg}(R), \phi\colon \widetilde{\mathcal{A}} \otimes_A k \simeq \mathcal{A} \right\}$$

Then, define $\mathrm{Der}^{\mathrm{E_n}}(\mathcal{A})$ to be the shifted tangent Lie algebra $T^{\mathrm{Lie}}[-1]_{\mathrm{E}_n\text{-alg}_{\widehat{\mathcal{A}}}}$. (This pre-fmp is defined on E_{n+1}-algebras, so that this Lie algebra structures comes from a E_{n+1}-algebra structure on on $T^{\mathrm{E}_{n+1}}[-n-1]_{\mathrm{E}_n\text{-alg}_{\widehat{\mathcal{A}}}}$.)

7.4.2 Infinitesimal Lie-, P_n-, Calc_n-adjoint actions

7.4.2.1. Suppose L is a dg Lie algebra. Then, there is a map of dg Lie algebras $L \to \mathrm{Der}^{\mathrm{Lie}}(L)$ given by $a \mapsto [a, -]$.

This has an interpretation in terms of formal moduli problems as follows: Consider the inclusion of constant loops $s\colon BG_L \to L(BG_L)$ along with the projection $p\colon L(BG_L) \to BG_L$. Then, the relative tangent complex $T_s \in QC(BG_L) \simeq (k\text{-mod})^L$ carries the structure of a Lie algebra: This is the adjoint representation L^{ad} as a Lie algebra in Lie-modules over L.

7.4.2.2. Suppose (L, M) is a $\mathrm{Lie}[+1]^+_\delta$-algebra. Recall that this means that $L[+1]$ is a dg Lie algebra and that $M[+1]$ is a dg Lie module for $k[+1] \oplus L[+1]$. Then, there is a map of dg Lie algebras $L[+1] \to \mathrm{Der}^{\mathrm{Lie}[+1]^+_\delta}(L, M)$ given by $a \mapsto [a, -]$ on L and $a \mapsto [a, -]$ on M.

7.4.2.3. Suppose $\mathscr{G}$ is a P_n-algebra. Then, there is a map of dg Lie algebras $\mathscr{G}[n-1] \to \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G})$ given by $a \mapsto [a, -]$. We will see below a model of this for a bigger model of P_2, the operad G_∞, in §7.5.2.

7.4.2.4. Suppose $(\mathscr{G}, M)$ is a Calc_n-algebra. Then, there is a map of dg Lie algebra $\mathscr{G}[n-1] \to \mathrm{Der}^{\mathrm{Calc_n}}(\mathscr{G}, M)$ given by $a \mapsto [a, -]$ on $\mathscr{G}$, and $a \mapsto [a, -]$ on M.

Remark 7.4.2.5. For us, these formulas will be interpreted as giving maps of colored operads. e.g., for the P_n adjoint action we have:

- There is the 2-colored dg operad $\mathrm{Lie}[n-1] \ltimes P_n$ whose algebras are pairs $(L, \mathscr{G})$ such that $L[n-1]$ is a dg Lie algebra and $\mathscr{G}$ is a P_n-algebra in dg Lie modules for $L[n-1]$. (Or, equivalently, $L[n-1]$ is a Lie algebra, $\mathscr{G}$ is a P_n-algebra, and there is given a map $L[n-1] \to \mathrm{Der}^{\mathrm{P_n}}(\mathscr{G})$.)

- The formula above determines a factorization

$$\begin{array}{ccc} \mathrm{Lie}[n-1] & \longrightarrow & P_n \\ & \searrow & \overset{ad}{\nearrow} \\ & \mathrm{Lie}[n-1] \ltimes P_n & \end{array}$$

  of the map $\mathrm{Lie}[n-1] \to P_n$.

7.4.3 Infinitesimal E_n adjoint action

Suppose that $\mathcal{C}$ is a k-linear E_n-monoidal $(\infty,1)$-category, and that $\mathcal{A}$ is an E_n-algebra in $\mathcal{C}$. The *infinitesimal adjoint action* will be a lift of $\mathcal{A}$ to an E_n-algebra in Lie-modules over the Lie algebra $\mathcal{A}[n-1]$. Or, what is the same, a map of Lie algebras $\mathcal{A}[n-1] \longrightarrow \mathrm{Der}^{\mathrm{E_n}}(\mathcal{A})$.

7.4.3.1. In light of the above, to define the infinitesimal adjoint action we just imitate the moduli-theoretic description from before. We must verify that there is a map of pre-formal moduli problems

$$\mathrm{End}^n_{/\mathcal{A}} \colon B^n\Omega^n\left(\widehat{\mathrm{mod}^n\text{-}\mathcal{A}}^{pre}\right) \longrightarrow \mathrm{E}_n\text{-}\mathrm{alg}^{pre}_{\widehat{\mathcal{A}}}$$

In light of the functoriality 7.3.3.3, it suffices to check the following: Suppose $A \to A'$ is a map in **DArt**, and $M \in [\mathrm{mod}^n\text{-}(\mathcal{A} \otimes A)]$ is equivalent to $\mathrm{mod}^{n-1}\text{-}(\mathcal{A} \otimes A)$, then the natural map

$$\mathrm{End}^n_{/\mathcal{A}\otimes A}(M) \otimes_A A' \longrightarrow \mathrm{End}^n_{/\mathcal{A}\otimes A'}\left(M \otimes_A A'\right)$$

is an equivalence. Similarly, §7.3.5 gives map of pre-fmp giving the infinitesimal E_n^{calc} adjoint action.

7.4.3.2. Since the Lie theoretic definitions above did not require any unitality, one might expect the same here. Suppose that $\mathfrak{m}_A$ is a non-unital E_n-algebra. Imitating the constructions of §7.3.4, one obtains a map of Lie algebras $(k \oplus \mathfrak{m}_{\mathcal{A}})[n-1] \to \mathrm{Der}^{\mathrm{E_n}}(\mathfrak{m}_{\mathcal{A}})$ together with a trivialization of the restriction to $k[n-1]$: This gives to the *non-unital infinitesimal* E_n *adjoint action* desired morphism $\mathfrak{m}_{\mathcal{A}}[n-1] \to \mathrm{Der}^{\mathrm{E_n}}(\mathfrak{m}_{\mathcal{A}})$.

7.4.4 Calculus-type adjoint actions variant

7.4.4.1. The functor of §7.3.3 admits a refinement as follows. Let $(\mathcal{A}, M) \in \mathrm{E}_n^{calc}$-alg, then:

- Consider the E_n-monoidal category $\mathcal{A}\text{-mod}^{\mathrm{E}_n}$ of operadic $\mathcal{A}$-modules. Since $\mathcal{A}$ is the monoidal unit, it is certainly an E_n-algebra and M is certainly a module over it, with prescribed equivariance. In other words, we can lift $(\mathcal{A}, M)$ to $(\mathcal{A}, M)^{\mathrm{ad}} \in \mathrm{E}_n^{calc}\text{-alg}(\mathcal{A}\text{-mod}^{\mathrm{E}_n})$ with the caveat that the notion of E_n^{calc}-algebra must take into account the $\mathrm{SO}(n)$-action on the category. The sense in which this refines the previous construction is:

- There is an $\mathrm{SO}(n)$-equivariant lax monoidal functor $F \colon \mathcal{A}\text{-mod}^{\mathrm{E}_n} \to \mathcal{A}[n-1]\text{-mod}^{\mathrm{Lie}}$ determined by an $\mathrm{SO}(n)$-equivariant map of Lie algebras $\mathcal{A}[n-1] \to U^{\mathrm{E}_n}(\mathcal{A})^{\mathrm{Lie}}$. Identifying $U^{\mathrm{E}_n}(\mathcal{A})$ with the topological chiral homology $\int_{S^{n-1}} \mathcal{A}$ (recall that while S^{n-1} is not framed, it is n-framed), one should presumably be able to produce this map by lifting $\mathcal{A}^{\mathrm{Lie}} \in \mathrm{E}_{n-1}\text{-alg(Lie-alg)}$ to $\mathrm{SO}(n-1)$-invariants, taking topological chiral homology on S^{n-1}, and composing with the fundamental class of S^{n-1}.

- Now, taking the image of $(\mathcal{A}, M)^{\mathrm{ad}} \in \mathrm{E}_n^{calc}\text{-alg}(\mathcal{A}\text{-mod}^{\mathrm{E}_n})$ under this lax monoidal functor produces $F\left((\mathcal{A}, M)^{\mathrm{ad}}\right) \in \mathrm{E}_n^{calc}\text{-alg}(\mathcal{A}\text{-mod}^{\mathrm{E}_n})$.

7.4.5 Universality for $n = 2$

We now prove Theorem 7.1.2.1, that the action of $\mathbf{HH}^\bullet(\mathcal{C})[+1]$ on $\mathcal{A} = (\mathbf{HH}^\bullet(\mathcal{C}), \mathbf{HH}_\bullet(\mathcal{C})) \in \mathrm{E}_2^{calc}$-alg only depends on $\mathcal{A}$ itself.

Proof of Theorem 7.1.2.1. The same as Theorem 7.1.2.1, just with formal moduli problems instead of spaces. □

7.5 Making things explicit in the case $k = 2$

7.5.1 Explicit model for E_2: Bialgebras and B_∞-algebras

Definition 7.5.1.1. A B_∞-structure on a complex $\mathcal{A}$ is a dg-bialgebra structure on $\mathrm{Bar}(\mathcal{A}) \stackrel{\text{def}}{=} \mathrm{coFree}^{\mathrm{coAss}}(\mathcal{A}[+1])$ equipped with its usual coproduct. It is a model for the E_2-operad, which while not cofibrant, is good enough in the following sense: If one begins with the (ordinary) category of B_∞-algebras and inverts quasi-isomorphisms, one obtains the ∞-category E_2-alg(k-mod).

7.5.1.2. The inclusion $\mathrm{E}_1 \hookrightarrow \mathrm{E}_2$ can be modeled by a map $A_\infty \to B_\infty$ as follows: Recall that an A_∞-structure on $\mathcal{A}$ is a dg-coalgebra structure on $\mathrm{Bar}(\mathcal{A})$ equipped with its usual coproduct, so one obtains a map by forgetting the extra product on $\mathrm{Bar}(\mathcal{A})$.

Lemma 7.5.1.3. *Suppose $(\mathcal{H}, d, m, \Delta)$ is a dg-bialgebra. Let $[,]\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ be the graded commutator $[x, x'] = m(x, x') - (-1)^{|x||x'|} m(x', x)$. Set*

$$\mathrm{Prim}(\mathcal{H}) = \{a \in \mathcal{H} \colon \Delta(a) = a \otimes 1 + 1 \otimes a\}$$

Then,

(i) $\mathrm{Prim}(\mathcal{H})$ *is preserved by the differential and closed under the graded commutator. The triple* $(\mathrm{Prim}(\mathcal{H}), d, [,])$ *is a dg Lie algebra.*

(ii) The restriction $[,]\colon \mathrm{Prim}(H) \otimes \mathcal{H} \to \mathcal{H}$ *makes $\mathcal{H}$ a Lie module over* $\mathrm{Prim}(\mathcal{H})$*.*

(iii) $(\mathcal{H}, d, m, \Delta)$ *is a dg-bialgebra in dg Lie-modules over* $\mathrm{Prim}(\mathcal{H})$*. (i.e., the action in (ii) is by derivations of the dg-bialgebra)*

7.5.1.4. The map $\mathrm{Lie}[+1] \to \mathrm{E}_2$ can be modelled by a map $\mathrm{Lie}[+1] \to B_\infty$ as follows: If $\mathcal{A}$ is a B_∞ algebra, then $\mathcal{A}[+1] \hookrightarrow \mathrm{Bar}(\mathcal{A})$ is the inclusion of primitives for the co-algebra structure. By the above Lemma(i), the bi-algebra structure thus equips $\mathcal{A}[+1]$ with the structure of dg Lie algebra. By the above Lemma(iii), $\mathrm{Bar}(\mathcal{A})$ is a dg-bialgebra in dg Lie modules over $\mathrm{Prim}(\mathcal{H})$—i.e., $\mathcal{A}$ is a B_∞-algebra in dg Lie modules over $\mathcal{A}[+1]$.

7.5.2 Partially resolved model for P_2: Lie bialgebras and G_∞-algebras

Definition 7.5.2.1. A G_∞-structure on a complex $\mathscr{G}$ is a dg Lie bialgebra structure on $\mathrm{Harr}(\mathscr{G}, k) \stackrel{\text{def}}{=} \mathrm{coFree}^{\mathrm{coLie}}(\mathscr{G}[+1])$ equipped with its usual cobracket. This is a model for the P_2-operad, which while not cofibrant, is good enough in the following sense: If one begins with the (ordinary) category of G_∞-algebras and inverts quasi-isomorphisms, one obtains the ∞-category P_2-alg(k-mod).

Lemma 7.5.2.2. *Suppose $(\mathfrak{g}, d, [], \delta)$ is a dg Lie bialgebra. Set*

$$\mathfrak{h} = \ker \delta = \{a \in \mathfrak{g} \colon \delta(a) = 0\}$$

Then,

(i) $\mathfrak{h}$ is preserved by the differential and the bracket. The triple $(\mathfrak{h}, d, [])$ is a dg Lie algebra.

(ii) The restriction $[]\colon \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$ makes $\mathfrak{g}$ a Lie module over $\mathfrak{h}$.

(iii) $(\mathfrak{g}, d, [], \delta)$ *is a dg Lie bialgebra in dg Lie modules over* $\mathfrak{h}$*. (i.e., the action in (ii) is by derivations of the dg Lie bialgebra).*

7.5.2.3. The map $\mathrm{Lie}[+1] \to P_2$ can be modelled by a map $\mathrm{Lie}[+1] \to G_\infty$ as follows: If $\mathscr{G}$ is a G_∞ algebra, then $\mathscr{G}[+1] \hookrightarrow \mathrm{Harr}(\mathscr{G}, k)$ is the inclusion of $\ker \delta$ for the Lie co-algebra structure. By the above Lemma(i), the Lie bi-algebra structure thus equips $\mathscr{G}[+1]$ with the structure of dg Lie algebra. By the above Lemma(iii), $\mathrm{Harr}(\mathcal{A}, k)$ is a dg Lie bialgebra in dg Lie modules over $\ker \delta$—i.e., $\mathscr{G}$ is a G_∞-algebra in dg Lie modules over $\mathscr{G}[+1]$.

7.5.3 Etingof-Kazhdan Quantization: From Lie bialgebras to Hopf algebras

7.5.3.1. We sketch a description of DQ_Φ^{-1} using Etingof-Kazhdan's theory of quantization of Lie bialgebra. We largely follow the exposition in [H]. It will be more convenient to work with coalgebras everywhere.

7.5.3.2. The procedure will go from co-G_∞-coalgebra structures on a graded vector space V to co-B_∞-coalgebra structures on V as follows:

– Etingof-Kazhdan guarantee an equivalence of categories

$$\left\{ \begin{matrix} k[\![\hbar]\!]\text{-linear Lie bialgebras} \\ \text{whose cobracket vanishes mod } \hbar \end{matrix} \right\} \xrightarrow{Q} \left\{ \begin{matrix} k[\![\hbar]\!]\text{-linear Hopf algebras} \\ \text{deforming } U(\mathfrak{g}) \text{ for some } \mathfrak{g} \end{matrix} \right\}$$

such that $Q(\mathfrak{g}) \otimes_{k[\![\hbar]\!]} k = U(\mathfrak{g} \otimes_{k[\![]\!]} k)$, $\delta \equiv \Delta - \Delta^{op} \pmod{\hbar^2}$, and Q is given by universal formulas in $[]$ and $\hbar\delta$ on $\mathfrak{g}$ not using $\hbar$.

– Begin with a co-G_∞ structure on V. That is, a dg Lie bialgebra structure $\left(\widehat{\mathrm{Free}^{\mathrm{Lie}}}(V[+1]), d, \delta, []\right)$ equipped with the usual free $[,]$. Passing from a filtered object to its Rees construction, we will instead work with $\mathfrak{g} = \left(\mathrm{Free}^{\mathrm{Lie}}(V[+1])[\![\hbar]\!], d, \delta, []\right)$ as graded $k[\![\hbar]\!]$-module with $\deg \hbar = 1$.

– Apply Q to this dg Lie bialgebra to obtain an equivariant dg bialgebra

$$Q(\mathfrak{g}) = Q\left(\mathrm{Free}^{\mathrm{Lie}}(V[+1])[\![\hbar]\!], d, \delta, []\right) = \left(\mathrm{Sym}(\mathrm{Free}^{\mathrm{Lie}}(V[+1]))[\![\hbar]\!], d_{sym}, m_{\delta,[]}, \Delta_{\delta,[]}\right)$$

Now, the inclusion $V[+1] \to \mathrm{Free}^{\mathrm{Lie}}(V[+1]) \to Q(\mathfrak{g})$ induces a map of graded dg algebras over $k[\![\hbar]\!]$

$$\Phi = \Phi_{\delta,[]} \colon \mathrm{Free}^{\mathrm{Ass}}(V[+1])[\![\hbar]\!] \to Q(\mathfrak{g})$$

– The above implies that $\Phi \otimes_{k[\![\hbar]\!]} k$ is the identity, so that this is an isomorphism of dg algebras. Then,

$$\left(\mathrm{Free}^{\mathrm{Ass}}(V[+1])[\![\hbar]\!], \Phi^*_{\delta,[]} d_{sym}, m_{Free}, \Phi^*_{\delta,[]} \Delta_{\delta,[]}\right)$$

is a $k[\![\hbar]\!]$-linear graded algebra having the standard free product. This determines a co-B_∞-coalgebra structure on V.

This explicit description, for instance, allows us to deduce the compatibility of the E_2 and P_2 adjoint actions under the passage to associated gradeds:

Proof of Prop. 7.1.3.2. We first claim that the associated graded of

$$\mathrm{Lie}[+1] \ltimes B_\infty \overset{\mathrm{ad}_{\mathrm{E}_2}}{\to} B_\infty$$

is (on the nose)

$$\mathrm{Lie}[+1] \ltimes G_\infty \overset{\mathrm{ad}_{P_2}}{\to} G_\infty$$

Tracing through the above construction, this is be a consequence of the condition that $\delta \equiv \Delta - \Delta^{op} \pmod{\hbar^2}$.

The fact that

$$\mathrm{gr}\left(\mathrm{Lie}[+1] \ltimes \mathrm{E}_2^{calc} \overset{\mathrm{ad}_{\mathrm{E}_2}}{\to} \mathrm{E}_2^{calc}\right) = \mathrm{Lie}[+1] \ltimes \mathrm{Calc}_2 \overset{\mathrm{ad}_{\mathrm{E}_2}}{\to} \mathrm{E}_2^{calc}$$

will then follow, since we already know that the associated graded of the Lie algebra and its module structure is correct. □

7.5.4 Compatibility of Lie and E_2 adjoint actions

Theorem 7.5.4.1. *Let* $\phi\colon \mathrm{Lie}[+1] \to \mathrm{E}_2$, $\phi'\colon \mathrm{Lie}[+1]^+_\delta \to \mathrm{E}_2^{calc}$ *be the usual maps. Then, the diagrams of colored operads*

$$\begin{array}{ccc} \mathrm{Lie}[+1] \ltimes \mathrm{Lie}[+1] & \xrightarrow{\mathrm{ad}_{\mathrm{Lie}[+1]}} & \mathrm{Lie}[+1] \\ \downarrow{\scriptstyle \mathrm{id} \times \phi} & & \downarrow{\scriptstyle \phi} \\ \mathrm{Lie}[+1] \ltimes \mathrm{E}_2 & \xrightarrow{\mathrm{ad}_{\mathrm{E}_2}} & \mathrm{E}_2 \end{array}$$

$$\begin{array}{ccc} \mathrm{Lie}[+1] \ltimes \mathrm{Lie}[+1]^+_\delta & \xrightarrow{\mathrm{ad}_{\mathrm{Lie}[+1]^+_\delta}} & \mathrm{Lie}[+1]^+_\delta \\ \downarrow{\scriptstyle \mathrm{id} \times \phi'} & & \downarrow{\scriptstyle \phi'} \\ \mathrm{Lie}[+1] \ltimes \mathrm{E}_2^{calc} & \xrightarrow{\mathrm{ad}_{\mathrm{E}_2}} & \mathrm{E}_2^{calc} \end{array}$$

are homotopy commutative.

Proof. In the B_∞ model given above, this diagram is strictly commutative. The $\mathrm{Lie}[+1]^+_\delta$ version then follows since we already knew that the underlying Lie algebra, its module, and δ were compatible with passage to associated gradeds (by looking at where $\mathrm{Lie}[+1]^+_\delta \to \mathrm{E}_2^{calc}$ sits in the filtration). □

Appendix A

Ind-Coherent Complexes

A.1 Descent for $\mathrm{QC}^!$ and MF^∞

A.1.1 Preliminaries

We need the following standard local-to-global tool:

Lemma A.1.1.1. *Suppose $\pi \colon U \to X$ is a flat map between ($\star$) derived stacks.*

(i) For $\mathscr{F}, \mathscr{G} \in \mathrm{QC}(X)$ there is a natural map

$$\pi^* \mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{RH}om^{\otimes U}_{\mathrm{QC}(U)}(\pi^*\mathscr{F}, \pi^*\mathscr{G}) \in \mathrm{QC}(U)$$

which is an equivalence provided that either

- *$\mathscr{F} \in \mathrm{Perf}(X)$ and $\mathscr{G}$ is arbitrary; or,*
- *$\mathscr{F}$ pseudo-coherent, and $\mathscr{G}$ is (locally) bounded above.*

(ii) For $\mathscr{F}, \mathscr{G} \in \operatorname{Ind}\mathrm{DCoh}(X)$ there is a natural map

$$\pi^* \mathcal{RH}om^{\otimes X}_{\operatorname{Ind}\mathrm{DCoh}(X)}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{RH}om^{\otimes U}_{\operatorname{Ind}\mathrm{DCoh}(U)}(\pi^*\mathscr{F}, \pi^*\mathscr{G}) \in \mathrm{QC}(U)$$

which is an equivalence provided that $\mathscr{F} \in \mathrm{DCoh}(X)$ and $\mathscr{G}$ is arbitrary.

Proof.

(i) The map is adjoint to a morphism

$$\mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}) \to \pi_* \mathcal{RH}om^{\otimes U}_{\mathrm{QC}(U)}(\pi^*\mathscr{F}, \pi^*\mathscr{G}) = \mathcal{RH}om^{\otimes X}_{\mathrm{QC}(U)}(\pi^*\mathscr{F}, \pi^*\mathscr{G})$$

characterized by the mapping property

$$\mathrm{Map}_{\mathrm{QC}(X)}(T \otimes \mathscr{F}, \mathscr{G}) \xrightarrow{\pi^*} \mathrm{Map}_{\mathrm{QC}(U)}(\pi^* T \otimes \pi^*\mathscr{F}, \pi^*\mathscr{G})$$

If $\mathscr{F}$ is perfect, then

$$\mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}) = \mathscr{F}^\vee \otimes \mathscr{G}$$

so that the claim is immediate from π^* being symmetric-monoidal.

Next suppose that $\mathscr{F}$ is pseudo-coherent and $\mathscr{G}$ (locally) bounded above: We first reduce to the case of U and X affine.[1] The affine case implies that

$$[p\colon \operatorname{Spec} A \to X] \mapsto \mathrm{RHom}^{\otimes A}(p^*\mathscr{F}, p^*\mathscr{G})$$

is a Cartesian section, i.e., lies in $\lim_{\mathrm{Aff}^\flat/X} \mathrm{QC}(A) = \mathrm{QC}(X)$ where the last equality is by faithfully flat descent. Since tensor product commutes with pullback, one readily checks that it satisfies the universal property characterizing $\mathrm{RHom}^{\otimes X}(\mathscr{F}, \mathscr{G})$. By faithfully flat descent, and since colimits of sheaves are preserved under fiber products, it suffices to prove that the map is an equivalence after further pullback along each fppf map $q\colon \operatorname{Spec} B \to U$ admitting a lift $\pi'\colon \operatorname{Spec} B \to \operatorname{Spec} A$ of π; that is, we must check that the natural map

$$q^*\pi^* \mathrm{RHom}^{\otimes X}(\mathscr{F}, \mathscr{G}) \longrightarrow q^* \mathrm{RHom}^{\otimes U}(\pi^*\mathscr{F}, \pi^*\mathscr{G})$$

is an equivalence. But by the above (applied once to X, once to Y) we naturally identify both sides with $\mathrm{RHom}^{\otimes B}(\mathscr{F}|_B, \mathscr{G}|_B)$.

We may thus suppose $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$. Since X is affine, it is in particular quasi-compact so that (shifting if necessary) we may suppose that $\mathscr{F}$ is connective and that $\mathscr{G}$ is bounded above. Furthermore, since X is affine we may write $\mathscr{F} \simeq |P_\bullet|$ as the geometric realization of a diagram of finite free connective A-modules, $P_k \simeq A^{\oplus n_k}$. In this case, we may identify

$$\pi^*\mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}) = B \otimes_A \mathrm{Tot}\left\{P_\bullet^\vee \otimes_A \mathscr{G}\right\}$$

$$\mathcal{RH}om^{\otimes U}_{\mathrm{QC}(U)}(\pi^*\mathscr{F}, \pi^*\mathscr{G}) = \mathrm{Tot}\left\{B \otimes_A (P_\bullet^\vee \otimes_A \mathscr{G})\right\}$$

and it remains to verify that tensor in fact commuted with the Tot by computing homotopy groups using a Bousfield-Kan spectral sequence. Since B is flat over A, we see that $\pi_0 B \otimes_{\pi_0 A} -$ of the Bousfield-Kan spectral sequence for the first Tot identifies with the B-K spectral sequence for the second Tot, so that it suffices to prove that both are convergent. Noting that $P_k^\vee \otimes_A \mathscr{G} = (A^{\oplus n_k})^\vee \otimes_A \mathscr{G} \simeq \mathscr{G}^{\oplus n_k}$ has homotopy groups in the same degrees as $\mathscr{G}$, and the same after the flat base extension $B \otimes_A -$, we readily conclude that they are convergent: If N is such that $\tau_{\geq N}\mathscr{G} = 0$, then the only terms that contribute to $\pi_\ell \mathrm{Tot}$ are ($\pi_0 B \otimes_{\pi_0 A} -$ of) $\pi_{q+\ell}(P_q^\vee \otimes_A \mathscr{G}) \simeq \pi_{q+\ell}(\mathscr{G})^{\oplus n_q}$ for $1 \leq q < N$.

(ii) The map is constructed analogously to that in (i) above. Note that π^* always preserves pseudo-coherent, and here it preserves (local) boundedness since π is flat. So, if $\mathscr{F} \in \mathrm{DCoh}(X)$ then $\pi^*\mathscr{F} \in \mathrm{DCoh}(X)$. Suppose now that $\mathscr{F} \in \mathrm{DCoh}(X)$, and $\mathscr{G} =$ "$\varinjlim_\beta$"$\mathscr{G}_\beta \in \operatorname{Ind}\mathrm{DCoh}(X)$. We claim that there is a natural equivalence

$$\mathcal{RH}om^{\otimes X}_{\operatorname{Ind}\mathrm{DCoh}(X)}(\mathscr{F}, \mathscr{G}) = \varinjlim_\beta \mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}_\beta)$$

[1] This reduction, and thus (i), does not actually require the hypothesis ($\star$).↑

(and similarly on U). Indeed for $T \in \mathrm{Perf}(X)$ we have $T \otimes \mathscr{F} \in \mathrm{DCoh}(X)$, so that

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QC}(X)}(T, \mathcal{RH}om^{\otimes}_{\mathrm{Ind\,DCoh}(X)}(\mathscr{F}, \mathscr{G})) &= \mathrm{Map}_{\mathrm{Ind\,DCoh}(X)}(T \otimes \mathscr{F}, \mathscr{G}) \\
&= \varinjlim_{\beta} \mathrm{Map}_{\mathrm{DCoh}(X)}(T \otimes \mathscr{F}, \mathscr{G}_\beta) \\
&= \varinjlim_{\beta} \mathrm{Map}_{\mathrm{QC}(X)}(T, \mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}_\beta)) \\
&= \mathrm{Map}_{\mathrm{QC}(X)}(T, \varinjlim_{\beta} \mathcal{RH}om^{\otimes X}_{\mathrm{QC}(X)}(\mathscr{F}, \mathscr{G}_\beta))
\end{aligned}$$

since T is compact. This reduces us to the case $\mathscr{F}, \mathscr{G} \in \mathrm{DCoh}(X)$, which follows from the second point of the QC case. □

A.1.2 Descent for $\mathrm{QC}^!$

Definition A.1.2.1. Suppose X is a derived scheme (or stack). Let X_{et} (resp., X_{sm}) denote the (small) site of morphisms $f: U \to X$ such that f is representable, bounded, and étale (resp., smooth); covers are defined as usual (i.e., surjectivity on geometric points). Note that any morphism between objects of X_{et} (resp., X_{sm}) is étale (resp., of finite Tor dimension). By Nisnevich descent, we mean descent for the Grothendieck topology generated by Nisnevich distinguished squares.

A.1.2.2. Recall that all morphisms in X_{sm} are of finite Tor-dimension. So, it makes sense to consider Ind DCoh as a pre-sheaf on X_{sm} via *star pullbacks*:

$$U \mapsto \mathrm{Ind\,DCoh}(U), \qquad f: U' \to U \mapsto f^*: \mathrm{Ind\,DCoh}(U) \to \mathrm{Ind\,DCoh}(U')$$

We will do this *only for the next proposition*, elsewhere we will use shriek pullbacks.

Proposition A.1.2.3. *Suppose that X is an ($\star$) derived stack, and that $\pi = \pi_\bullet : U_\bullet \to X$ is a smooth hypercover. Then, the natural functor*

$$\pi^*: \mathrm{Ind\,DCoh}(X) \longrightarrow \mathrm{Tot}\left\{\mathrm{Ind\,DCoh}(U_\bullet), \pi_\bullet^*\right\}$$

to the descent category is fully faithful. Consequently, the natural functor

$$\pi^!: \mathrm{Ind\,DCoh}(X) \longrightarrow \mathrm{Tot}\left\{\mathrm{Ind\,DCoh}(U_\bullet), \pi_\bullet^!\right\}$$

is also fully faithful.

Proof. We must show that the natural map

$$\mathrm{Map}_{\mathrm{Ind\,DCoh}\,X}\left(\text{“}\varinjlim_{\alpha}\text{”}\mathscr{F}_\alpha, \text{“}\varinjlim_{\beta}\text{”}\mathscr{G}_\beta\right) \longrightarrow \mathrm{Tot}_\bullet \mathrm{Map}_{\mathrm{Ind\,DCoh}\,U_\bullet}\left(\text{“}\varinjlim_{\alpha}\text{”}\pi_\bullet^*\mathscr{F}_\alpha, \text{“}\varinjlim_{\beta}\text{”}\pi_\bullet^*\mathscr{G}_\beta\right)$$

is an equivalence. Since homotopy limits commute, we may reduce to the case of no α, i.e., just mapping out of $\mathscr{F} \in \mathrm{DCoh}(X)$.

Suppose $\mathscr{F} \in \mathrm{DCoh}(X)$, and "$\varinjlim_\beta$"$\mathscr{G}_\beta \in \mathrm{Ind}\,\mathrm{DCoh}(X)$. Set $U_{-1} = X$, and

$$\mathcal{RH}om_n \stackrel{\text{def}}{=} \mathcal{RH}om^{\otimes U_n}_{\mathrm{Ind}\,\mathrm{DCoh}\,U_n}\left(\mathscr{F}|_{U_n}, \text{“}\varinjlim_\beta\text{''}\, \mathscr{G}_\beta|_{U_n}\right)$$

for $n \geq -1$. By Lemma A.1.1.1, the natural map

$$\left[\mathcal{RH}om^{\otimes X}_{\mathrm{QC}^!(X)}\left(\mathscr{F}, \text{“}\varinjlim_\beta\text{''}\mathscr{G}_\beta\right)\right]\Bigg|_{U_n} \longrightarrow \mathcal{RH}om^{\otimes U_n}_{\mathrm{QC}^!(U_n)}\left(\mathscr{F}|_{U_n}, \text{“}\varinjlim_\beta\text{''}\, \mathscr{G}_\beta|_{U_n}\right)$$

is an equivalence. Now, fppf descent for QC implies

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}\,\mathrm{DCoh}\,X}\left(\mathscr{F}, \text{“}\varinjlim_\beta\text{''}\mathscr{G}_\beta\right) &= \mathrm{Map}_{\mathrm{QC}(X)}(\mathcal{O}_X, \mathcal{RH}om_{-1}) \\
&= \mathrm{Tot}_\bullet \mathrm{Map}_{\mathrm{QC}(U_n)}\left(\mathcal{O}_X|_{U_n}, \mathcal{RH}om_{-1}|_{U_n}\right) \\
&= \mathrm{Tot}_\bullet \mathrm{Map}_{\mathrm{QC}(U_n)}(\mathcal{O}_{U_n}, \mathcal{RH}om_n) \\
&= \mathrm{Tot}_\bullet \mathrm{Map}_{\mathrm{Ind}\,\mathrm{DCoh}\,U_n}\left(\mathscr{F}|_{U_n}, \text{“}\varinjlim_\beta\text{''}\, \mathscr{G}_\beta|_{U_n}\right)
\end{aligned}$$

as desired.

Finally, it remains to prove the "Consequently": Note that the totalization of a cosimplicial object coincides with the homotopy limit over its underlying *semi-cosimplicial* object. All the morphisms in the semi-simplicial object underlying $U_\bullet$ are smooth, so that there is an equivalence $f^!(-) = f^*(-) \otimes_{\mathcal{O}} \omega_f$ and the relative dualizing complex $\omega_f \simeq \det \mathbb{L}_f$ is invertible. Let $\omega_{\pi_\bullet} = \pi^!(\mathcal{O}_X)$, regarded as an invertible object in $\mathrm{Tot}\left\{\mathrm{QC}(U_\bullet)^\otimes, f^*\right\}$. Then, $\pi^!(-) \simeq \pi^*(-) \otimes_{\mathcal{O}_{U_\bullet}} \omega_{\pi_\bullet}$ is fully-faithful since both π^* and tensoring by an invertible object are fully-faithful. $\square$

Lemma A.1.2.4. *Suppose $f\colon X' \to X$ is a map of ($\star$) derived stacks. Then,*

(i) Suppose f is surjective (on field valued points) and proper with $(f_*, f^!)$ an adjoint pair after any base-change (e.g., finite).[2] If $\mathrm{Ind}\,\mathrm{DCoh}(-)$ is a sheaf on X'_{sm}, then it is a sheaf on X_{sm}.

(ii) Suppose that f satisfies the conditions of (ii). Then, there is a natural equivalence

$$\mathrm{Ind}\,\mathrm{DCoh}(X) = (q^! q_*)\text{-mod}\left(\mathrm{Ind}\,\mathrm{DCoh}(X')\right)$$

Proof.

(i) First note the property of f being proper and surjective is stable under base-change. Consequently, it suffices to show if $\pi = \pi_\bullet\colon U_\bullet \to X$ is a smooth hypercover of X itself, then the functor $\pi^*\colon \mathrm{Ind}\,\mathrm{DCoh}(X) \to \mathrm{Tot}\left\{\mathrm{Ind}\,\mathrm{DCoh}\,U_\bullet\right\}$ to the descent category is an equivalence. By the previous Proposition, it suffices to show that π^* is essentially surjective. Since the totalization may be computed in Pr^L, viewing p^*

[2] More generally this condition is satisfied if f is a relative proper algebraic space, or (in char. 0) a relative proper DM stack.↑

as left adjoint to p_*, we note that π^* admits a right adjoint π_* which is explicitly given by $\pi_*(\mathscr{F}_\bullet) = \operatorname{Tot}\{(\pi_\bullet)_*\mathscr{F}_\bullet\}$. It suffices to show that the counit $\pi^*\pi_* \to \mathrm{id}$ is an equivalence. Since the left-adjoint π^* is fully faithful, the unit map $\mathrm{id} \to \pi_*\pi^*$ is an equivalence; so, it suffices to show that π_* is conservative: Indeed, consider the factorization of $\mathrm{id}_{\pi_*\mathscr{F}}$ as

$$\pi_*\mathscr{F} \xrightarrow{\sim} \pi_*\pi^*\pi_*\mathscr{F} \xrightarrow{\pi_*(\text{counit})} \pi_*\mathscr{F}$$

Since the categories involved are stable, and the functors exacts, it suffices to show the following: If $\pi_*(\mathscr{F}_\bullet) = 0$, then $\mathscr{F}_\bullet = 0$; since $\mathscr{F}_n$, $n > 0$ is a pull-back of $\mathscr{F}_0$, it suffices to show that $\mathscr{F}_0 = 0$ under these hypotheses. So, we must prove that the functor

$$(\pi_0)^*\pi_* = (\pi_0)^* \operatorname{Tot}\{(\pi_n)_*\mathscr{F}_n\}$$

is conservative.

Let $U'_n = U_n \times_X X'$, $\pi'_\bullet$ the base-changed structure maps, and $f_n\colon U'_n \to U_n$ the first projection. It suffices to show that $(f_0)^! \circ (\pi_0)^* \circ \pi_*$ is conservative. By various standard compatibilities (which one, e.g., first checks on QC then extends to $\mathrm{QC}^!$ by t-bounded-above arguments):

$$(f_0)^!(\pi_0)^* \operatorname{Tot}\{(\pi_n)_*\mathscr{F}_n\} = (\pi'_0)^* f^! \operatorname{Tot}\{(\pi_n)_*\mathscr{F}_n\}$$

Since $f^!$ is a right adjoint, it commutes with arbitrary limits

$$\begin{aligned} &= (\pi'_0)^* \operatorname{Tot}\left\{f^!(\pi_n)_*\mathscr{F}_n\right\} \\ &= (\pi'_0)^* \operatorname{Tot}\left\{(\pi'_n)_*(f_n)^!\mathscr{F}_n\right\} = (\pi'_0)^*(\pi'_\bullet)_*((f_\bullet)^!\mathscr{F}_\bullet) \end{aligned}$$

If this vanishes, then (by the hypothesis on X') we find that $f_0^!\mathscr{F}_0 = 0$. So, it suffices to show that $(f_0)^!\colon \operatorname{Ind}\operatorname{DCoh} U_0 \to \operatorname{Ind}\operatorname{DCoh} U'_0$ is conservative. Note that f_0 is, being a base-change of f, also finite and surjective so that it suffices to show the following: If $f\colon X' \to X$ is a proper (in the sense of the footnote), surjective, map of ($\star$) derived stacks, then $f^!$ is conservative. Given $\mathscr{F} \in \operatorname{Ind}\operatorname{DCoh}(X)$ such that $f^!\mathscr{F} = 0$, it suffices to show that $\operatorname{Map}_{\operatorname{Ind}\operatorname{DCoh}(X)}(\mathscr{K}, \mathscr{F}) = 0$ for all $\mathscr{K} \in \operatorname{DCoh}(X)$. By Prop. A.1.2.3, this is smooth local on X, so we may suppose $X = \operatorname{Spec} A$ is an affine derived scheme. Considering the diagrams

$$\begin{array}{ccc} \pi_0 X' & \longrightarrow & \pi_0 X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \operatorname{Ind}\operatorname{DCoh}(\pi_0 X') & \longleftarrow & \operatorname{Ind}\operatorname{DCoh}(\pi_0 X) \\ \uparrow & & \uparrow \\ \operatorname{Ind}\operatorname{DCoh}(X') & \longleftarrow & \operatorname{Ind}\operatorname{DCoh}(X) \end{array}$$

and noting that $\operatorname{Ind}\operatorname{DCoh}(X) \to \operatorname{Ind}\operatorname{DCoh}(\pi_0 X)$ is conservative by Lemma 2.2.0.2, we may reduce to the case of X and $X' = \operatorname{Spec} \pi_0 A$ discrete and in particular ordinary separated stacks. Since $\pi_0 A$ is Noetherian, Chow's Lemma for stacks [O1] shows that X' receives a proper surjection from a projective $\pi_0 A$-scheme. Thus, it suffices to prove the claim in case X' is a projective $\pi_0 A$-scheme; let $\mathcal{O}(1)$ be a relatively ample line bundle. Let T denote the smallest thick subcategory of $\mathrm{DCoh}(X)$ containing $f_* \mathrm{DCoh}(X')$. Since $f^!$ is right adjoint to f_*, it suffices to show that $T = \mathrm{DCoh}(X)$: Indeed, $\ker f^!$ is right-orthogonal to T.

Since X is quasi-compact, a t-structure argument shows that it suffices to show that the intersection of T with the heart $T^\heartsuit = T \cap \mathrm{DCoh}(X)^\heartsuit$ is all of $\mathrm{Coh}(X) = \mathrm{DCoh}(X)^\heartsuit$. By the usual form of devissage, noting that $T^\heartsuit$ is closed under direct summands, it suffices to show that for all $x \in X$ there is some $\mathscr{G}(x) \in \mathrm{DCoh}(X')$ such that $p_*\mathscr{G}(x) \in T^\heartsuit$ and has a non-zero fiber over x. Since f is surjective, we may take $x' \in X'$ lying over it, and set $\mathscr{G}(x) = \mathcal{O}_{\overline{x'}} \otimes \mathcal{O}(N)$ for $N \gg 0$ large enough so that $p_*\mathscr{G}(X) \in T^\heartsuit$: That such an N exists is Serre's Theorem. Note that $\mathscr{G}(x) \otimes \mathcal{O}_{X,x}$ is a non-zero, coherent, $\mathcal{O}_{X,x}$-module whence the fiber at x is non-zero by Nakayama's Lemma.

(ii) By the proof of (ii), we have seen that $f^!$ is conservative. Since it preserves all colimits, Lurie's Barr-Beck Theorem applies to prove the first equality. □

We are now ready to prove the main result of this subsection:

Theorem A.1.2.5. *Suppose X is a ($\star$) derived stack. Let* $\mathrm{Ind}\,\mathrm{DCoh}(-)$ *denote the pre-sheaf*

$$U \mapsto \mathrm{Ind}\,\mathrm{DCoh}(U), \qquad [f : U \to U'] \mapsto f^!$$

Then,

(i) $\mathrm{Ind}\,\mathrm{DCoh}(-)$ *has Nisnevich descent, and finite étale descent.*

(ii) $\mathrm{Ind}\,\mathrm{DCoh}(-)$ *has representable étale descent.*

(iii) Suppose furthermore X is a derived DM stack, then $\mathrm{Ind}\,\mathrm{DCoh}(-)$ *has smooth descent.*

(iv) Suppose furthermore X is a derived DM stack almost of finite-presentation over k. Then, $\mathrm{Ind}\,\mathrm{DCoh}(-)$ *has smooth descent and* $\mathrm{Ind}\,\mathrm{DCoh}(X)$ *coincides with* $\mathrm{QC}^!(X)$ *(as defined in Section 4.1).*

Proof.

(i) By Prop. A.1.2.3, the pullback map to the descent category is fully faithful for any smooth hypercover so that it suffices to check essential surjectivity.

Step 1: Finite étale covers.
Suppose $p\colon X' = U_0 \to X$ is an étale cover which is a *finite* morphism, and $\pi_\bullet\colon U_\bullet \to X$ the Cech nerve. Let $U'_\bullet = U_\bullet \times_X X'$, with $\pi'_\bullet\colon U'_\bullet \to X'$ the base-changed maps, and $p'_\bullet = U'_\bullet \to U_\bullet$ the projections. Suppose $\{\mathscr{F}_\bullet\}$ is such that $\mathrm{Tot}\{(\pi_n)_*\mathscr{F}_n\} = 0$; we must show that $\mathscr{F}_0 = 0$.

Note that π_0 is both finite and étale, so that $(\pi_0)^* = (\pi_0)^!$ preserves all limits since it is right-adjoint to $(\pi_0)_*$. Consequently, there are natural equivalences

$$\begin{aligned} 0 = (\pi_0)^* \mathrm{Tot}\{(\pi_n)_*\mathscr{F}_n\} &= \mathrm{Tot}\{(\pi_0)^*(\pi_n)_*\mathscr{F}_n\} \\ &= \mathrm{Tot}\left\{(\pi'_n)_*(p_n)^*\mathscr{F}_n\right\} \\ &= (\pi')_*(\pi')^*\mathscr{F}_0 \end{aligned}$$

Fully-faithfullness of $(\pi')^*$ implies that the unit $\mathscr{F}_0 \to (\pi')_*(\pi')^*\mathscr{F}_0$ is an equivalence, so that $\mathscr{F}_0 = 0$.

Step 2: Distinguished Nisnevich squares. Suppose given a distinguished Nisnevich square

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & X \end{array}$$

with j an open immersion, p étale. Let $Z = X \setminus U$ and $Z' = X' \setminus U'$. We must prove that

$$\pi^*\colon \operatorname{Ind}\operatorname{DCoh} X \longrightarrow \operatorname{Ind}\operatorname{DCoh} U \times_{\operatorname{Ind}\operatorname{DCoh} U'} \operatorname{Ind}\operatorname{DCoh} X'$$

is an equivalence. We know that π^* is fully faithful so that it suffices to prove essential surjectivity. Given the adjunction (π^*, π_*), it suffices to show that the counit $\pi^*\pi_* \to \mathrm{id}$ is an equivalence; since π^* is fully faithful, the unit $\mathrm{id} \to \pi_*\pi^*$ is an equivalence and considering the following factorization of the identity on $\pi_*\mathscr{F}$

$$\pi_*\mathscr{F} \xrightarrow{\sim} \pi_*\pi^*\pi_*\mathscr{F} \overset{\pi_*((counit)}{\longrightarrow} \pi_*\mathscr{F}$$

reduces us to showing that π_* is conservative. Since all categories involved are stable and π_* is exact, it suffices to prove that $\ker \pi_* = 0$. Suppose

$$\mathscr{F}_\star = (\mathscr{F}_U, \mathscr{F}_{U'}, \mathscr{F}_{X'}) \in \operatorname{Ind}\operatorname{DCoh} U \longrightarrow \operatorname{Ind}\operatorname{DCoh} U \times_{\operatorname{Ind}\operatorname{DCoh} U'} \operatorname{Ind}\operatorname{DCoh} X'$$

and recall that

$$\pi_*(\mathscr{F}_\star) = j_*\mathscr{F}_U \times_{j_*p'_*\mathscr{F}_{U'}} p_*\mathscr{F}_{X'} = j_*\mathscr{F}_U \times_{p_*(j')_*\mathscr{F}_{U'}} p_*\mathscr{F}_{X'} \in \operatorname{Ind}\operatorname{DCoh} X$$

It suffices to construct equivalences $j^*\pi_*\mathscr{F}_\star \simeq \mathscr{F}_U$ and $p^*\pi_*\mathscr{F}_\star = \mathscr{F}_{X'}$, for then $\pi_*\mathscr{F}_\star = 0$ implies $\mathscr{F}_U = 0$ and $\mathscr{F}_{X'} = 0$ (and so $\mathscr{F}_{U'} = 0$).

Note that the counit $j^*j_* \to \mathrm{id}$ is an equivalence and that there is a natural equivalence $j^*p_* = (p')_*(j')^*$: Both are true on QC and all functors involved are t-bounded-above. Consequently,

$$j^*\pi_*\mathscr{F}_\star = \mathscr{F}_U \times_{(p')_*\mathscr{F}_{U'}} (p')_*\mathscr{F}_{U'} = \mathscr{F}_U.$$

It remains to provide an equivalence $p^*\pi_*\mathscr{F}_\star = \mathscr{F}_{X'}$. First note that

$$\begin{aligned} p^*\pi_*\mathscr{F}_\star &= p^*\left(j_*(\mathscr{F}_U) \times_{p_*(j')_*\mathscr{F}_{U'}} p_*\mathscr{F}_{X'}\right) \\ &= p^*j_*(\mathscr{F}_U) \times_{p^*p_*(j')_*\mathscr{F}_{U'}} p^*p_*\mathscr{F}_{X'} \\ &= (j')_*\mathscr{F}_{U'} \times_{(j')_*(p')^*(p')_*\mathscr{F}_{U'}} p^*p_*\mathscr{F}_{X'} \end{aligned}$$

and from the last term we obtain a natural map $\phi\colon p^*\pi_*\mathscr{F}_\star \to \mathscr{F}_{X'}$ using the structure maps that the counit $p^*p_* \to \mathrm{id}$. Let $i\colon Z_{\mathrm{red}} \to X$ and $i'\colon Z'_{\mathrm{red}} \to X'$. By Lemma A.1.2.6 below, it suffices to show that $(j')^*\phi$ and $(i')^!\phi$ are equivalences. The former is straightforward (both sides naturally identity with $\mathscr{F}_{U'}$), as is the latter (since $p|_{Z'_{\mathrm{red}}} : Z'_{\mathrm{red}} \to Z_{\mathrm{red}}$ is an isomorphism by the definition of distinguished Nisnevich).

(ii) By Lemma A.1.2.4 we may reduce to the case of X discrete. Then, note that descent for distinguished Nisnevich squares and finite étale covers implies representable étale descent ([R, Theorem D, Remark 5.4]).

(iii) A derived DM stack with affine diagonal admits a representable étale cover by a scheme, so that every étale cover admits a representable refinement representable-étale locally. Since the cotangent complex of a derived DM stack is connective, any smooth cover admits an étale refinement.

(iv) By (iii) applied to $\operatorname{Spec} A$ for $A \in \mathbf{DRng}_k^{\mathrm{fp}}$, $\mathrm{QC}^!(-)$ is a smooth sheaf on X. Note that (iii) applies to X, so that $\operatorname{Ind}\operatorname{DCoh}(-)$ is a smooth sheaf on X. Since X was assumed DM and almost of finite-presentation, X is étale locally of the form $\operatorname{Spec} A$ for $a \in \mathbf{DRng}_k^{\mathrm{fp}}$ so that the two sheaves are locally isomorphic. □

Lemma A.1.2.6. *Suppose X is a ($\star$) derived stack, $j\colon U \subset X$ a quasi-compact open, and $i\colon Z_{\mathrm{red}} \to X$ the reduced-induced structure on the closed complement. Suppose $\phi\colon \mathscr{F} \to \mathscr{G}$ is a morphism in $\operatorname{Ind}\operatorname{DCoh}(X)$. Then, ϕ is an equivalence if and only if $j^*\phi$ and $i^!\phi$ are both equivalences.*

Proof. Taking $\operatorname{cone}(\phi)$, it suffices to show that $\mathscr{F} \in \operatorname{Ind}\operatorname{DCoh}(X)$ is zero iff $j^*\mathscr{F} = 0$ and $i^!\mathscr{F} = 0$. One direction is clear, so suppose $j^*\mathscr{F} = 0$ and $i^!\mathscr{F} = 0$. We must show that $\operatorname{Map}_{\operatorname{Ind}\operatorname{DCoh}(X)}(\mathscr{K}, \mathscr{F}) = 0$ for all $\mathscr{K} \in \operatorname{DCoh}(X)$. By Lemma A.1.1.1, it suffices to show that $\operatorname{RHom}^{\otimes X}_{\operatorname{Ind}\operatorname{DCoh}(X)}(\mathscr{K}, \mathscr{F}) = 0 \in \mathrm{QC}(X)$ and the question is fppf local on X by fppf descent for $\mathrm{QC}(X)$. In particular, we may assume that X is affine so that we are in the situation where we have sketched a proof of Lemma 4.1.4.2. It thus suffices to note that $i^!\colon \operatorname{Ind}\operatorname{DCoh}_Z(X) \to \operatorname{Ind}\operatorname{DCoh}(Z)$ is conservative, since its left-adjoint i_* hits a generating set by Lemma 2.2.0.2. □

Remark A.1.2.7. In fact, more is true than Theorem A.1.2.5. Let X_h denote the (derived) Grothendieck topology on representable, bounded, almost finitely-presented X-stacks generated by distinguished Nisnevich squares and proper (with $(f_*, f^!)$ adjunction) surjective maps.[3] The Theorem together with Prop. A.1.2.8 below imply that $\operatorname{Ind}\operatorname{DCoh}(-)$ has h-descent in this funny sense: Since X_h has covering morphisms which are not flat, the corresponding ∞-topos looks substantially different from the ordinary h-topos of $\pi_0 X$ even if X is a discrete affine scheme, e.g., the map $X_{\mathrm{red}} \to X$ is no longer a monomorphism, so that the natural map $\mathscr{F}(X) \to \mathscr{F}(X_{\mathrm{red}})$ need not be an equivalence.

Proposition A.1.2.8. *Suppose that X is a ($\star$) derived stack. Then,* $\operatorname{Ind}\operatorname{DCoh}(-)$ *has* proper *descent on X: i.e., Suppose $q\colon X' \to X$ is proper and surjective, and let $\pi = \pi_\bullet\colon \left\{X'_\bullet = (X')^{\bullet/X}\right\} \to X$ be the Cech nerve of q. Then, the functor*

$$\pi^!\colon \operatorname{Ind}\operatorname{DCoh}(X) \to \operatorname{Tot}\left\{\operatorname{Ind}\operatorname{DCoh}(X'_\bullet), f^!\right\}$$

is an equivalence of categories.

Proof. Note that all structure maps in $X'_\bullet$ are proper since $X' \to X$ is proper, and in particular separated; so, the totalization may be regarded as being computed in either Pr^R or Pr^L. Consequently $\pi^!$ admits a left-adjoint π_*. Consequently $\pi^!$ admits both preserves

[3] The naming is suggested by the fact that on an ordinary Noetherian scheme, the ordinary Grothendieck topology generated by the Nisnevich squares and proper surjections is precisely the ordinary h-topology.↑

colimits and admits a left-adjoint π_*. We can check that π_* is computed by the geometric realization

$$(\pi_* (\mathscr{F}_\bullet) = |(\pi_n)_*(\mathscr{F}_n)|$$

It now suffices to check that the unit and counit maps $\pi_*\pi^! \to \mathrm{id}$ and $\mathrm{id} \to \pi^!\pi_*$ are equivalences. Since $q^!$ is conservative by Lemma A.1.2.4, so is $\pi^!$. Thus it is enough to check that the unit map is an equivalence (c.f., Lemma 4.1.4.1(iv)).

For the unit: Since $q^!$ is conservative by Lemma A.1.2.4 it suffices to check this after applying $q^!$, so that we are interested in verifying that map

$$q^!\pi_*\pi^!\mathscr{F} = \left|q^!(\pi_n)_*(\pi_n)^!\mathscr{F}\right|\Big| \longrightarrow q^!\mathscr{F}$$

is an equivalence. Let $p_n \colon X'_{n+1} \to X'$ be the first projection (i.e., this is the base change of π_n along q), and $q_n \colon X'_{n+1} \to X_n$ the last projection (i.e., the induced map on simplicial objects). Base-change gives $q^!(\pi_n)_*(\pi_n)^! = (p_n)_*(q_n)^!(\pi_n)^!\mathscr{F} = (p_n)_*(\pi_{n+1})^!\mathscr{F}$, so that our augmented simplicial diagram is in fact *split* and consequently a colimit diagram. ☐

A.1.3 Descent for MF^∞

Proposition A.1.3.1. *Suppose X is a $(\star_{\mathrm{F}})$ derived stack, and $f \colon X \to \mathbb{A}^1$.*

(i) The assignments

$$U \mapsto \mathrm{PreMF}(U, f|_U) \qquad \textit{and} \qquad U \mapsto \mathrm{PreMF}^\infty(U, f|_U)$$

determine sheaves of $k[\![\beta]\!]$-linear ∞-categories on X_{et}.

(ii) The assignment

$$U \mapsto \mathrm{MF}^\infty(U, f|_U)$$

determines a sheaf of $k(\!(\beta)\!)$-linear ∞-categories on X_{et}.

Proof.

(i) Note that any étale cover of X_{et} restricts to an étale cover of X_0, and that a $k[\![\beta]\!]$-linear presheaf is a sheaf if and only if it is a sheaf forgetting the extra linear structure. So, it suffices to show that DCoh and $\mathrm{QC}^!$ are sheaves on $(X_0)_{et}$; the former follows from the analogous theorem for QC and the local definition of DCoh (since X_0 is coherent), and the latter follows from the analogous theorem for $\mathrm{QC}^!$ (Theorem A.1.2.5).

(ii) It suffices to note that $-\widehat{\otimes}_{k[\![\beta]\!]}k(\!(\beta)\!) \colon \mathbf{dgcat}^\infty_{k[\![\beta]\!]} \to \mathbf{dgcat}^\infty_{k(\!(\beta)\!)}$ commutes with homotopy-limits, since $k(\!(\beta)\!)\text{-mod} \in \mathbf{dgcat}^\infty_{k[\![\beta]\!]}$ is dualizable (c.f., Lemma 3.2.2.1). ☐

A.2 Integral transforms for (Ind) Coherent Complexes

We give here an exposition of the Tensor Product and Functor Theorems for $\mathrm{QC}^!$ of derived schemes. As this is essentially a mild generalization of [L1], we will be brief.

A.2.1 Fully faithful

Proposition A.2.1.1. *Suppose S is a regular $(\star)$ derived stack, and X and Y $(\star)$ derived stacks over S. Then, exterior product over S determines a well-defined and fully faithful* *functor*

$$\boxtimes_S \colon \operatorname{DCoh}(X) \otimes_S \operatorname{DCoh}(Y) \longrightarrow \operatorname{DCoh}(X \times_S Y)$$

Proof. We first check that it is well-defined. Since DCoh (with star pullback) is an fppf sheaf, the question is local on X and Y so that we may suppose $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R'$. Since S is assumed to have affine diagonal, X and Y are also affine over S so that the pushforward is t-exact, etc. Exterior product always preserves pseudo-coherence, and since S is regular we conclude that it preserves being locally bounded since we may check so after pushforward to S.

Next we check that it is fully-faithful, i.e., that the exterior product

$$\operatorname{RHom}^{\otimes S}_{\operatorname{QC}(X)}(\mathscr{F}_X, \mathscr{G}_X) \otimes_{\mathcal{O}_S} \operatorname{RHom}^{\otimes S}_{\operatorname{QC}(Y)}(\mathscr{F}_Y, \mathscr{G}_Y) \longrightarrow \operatorname{RHom}^{\otimes S}_{\operatorname{QC}(X \times_S Y)}(\mathscr{F}_X \boxtimes_S \mathscr{F}_Y, \mathscr{G}_X \boxtimes_S \mathscr{G}_Y)$$

is an equivalence in $\operatorname{QC}(S)$ for all $\mathscr{F}_X, \mathscr{G}_X \in \operatorname{DCoh}(X)$, and $\mathscr{F}_Y, \mathscr{G}_Y \in \operatorname{DCoh}(Y)$. By Lemma A.1.1.1, the claim is fppf local on X, Y and S, so that we may assume they are all affine: $S = \operatorname{Spec} A$, $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R'$.

The claim is clear when $\mathscr{F}_X = \mathcal{O}_X$ and $\mathscr{F}_Y = \mathcal{O}_Y$, and so more generally whenever $\mathscr{F}_X$ and $\mathscr{F}_Y$ are perfect. Shifting as necessary, we may suppose that $\mathscr{G}_X$, $\mathscr{G}_Y$, and $\mathscr{G}_X \boxtimes_S \mathscr{G}_Y$ are all co-connective (i.e., $\pi_i = 0$ for $i \geq 0$), while $\mathscr{F}_X$, $\mathscr{F}_Y$ (and hence $\mathscr{F}_X \boxtimes_S \mathscr{F}_Y$) are connective. Then, $\mathscr{F}_X$ (resp., $\mathscr{F}_Y$) can be written as the geometric realizations of a diagram of finite free R-modules $P_\bullet$ (resp., R'-modules $P'_\bullet$); since exterior product preserves colimits, $\mathscr{F}_X \boxtimes_S \mathscr{F}_Y$ will then be the realization of the bisimplicial object $P_\bullet \boxtimes_S P'_\bullet$. Under our connectivity assumptions, it is straightforward to check that the Bousfield-Kan spectral sequence for $\operatorname{RHom}_X(\mathscr{F}_X, \mathscr{G}_X) = \operatorname{Tot} \operatorname{RHom}_X(P_\bullet, \mathscr{G}_X)$ is convergent and that the RHom complex is co-connective:

$$E^1_{p,q} = \pi_p \operatorname{RHom}_X(P_q, \mathscr{G}_X) = (\pi_p \mathscr{G}_X)^{\oplus n_p} \Rightarrow \pi_{p-q} \operatorname{RHom}_X(\mathscr{F}_X, \mathscr{G}_X)$$

where n_p is the rank of the finite free R-module P_p. Similarly, the RHom complexes on Y and $X \times Y$ are also co-connective.

Next, note that that $- \otimes_S -$ commutes with totalizations of co-connective objects in each variables. This follows by another Bousfield-Kan spectral sequence, since S is regular so that it has bounded flat dimension.

Putting the above together, we may conclude

$$\begin{aligned}
\operatorname{RHom}_X(\mathscr{F}_X, \mathscr{G}_X) \otimes_S \operatorname{RHomy}_Y(\mathscr{F}_Y, \mathscr{G}_Y) &= (\operatorname{Tot}_a \operatorname{RHom}_X(\mathscr{P}_a, \mathscr{G}_X)) \otimes_S \left(\operatorname{Tot}_b \operatorname{RHomy}_Y(\mathscr{P}'_b, \mathscr{G}_Y)\right) \\
&= \operatorname{Tot}_{a,b} \left(\operatorname{RHom}_X(\mathscr{P}_a, \mathscr{G}_X) \otimes_S \operatorname{RHom}_Y(\mathscr{P}'_b, \mathscr{G}_Y)\right) \\
&= \operatorname{Tot}_{a,b} \operatorname{RHom}_{X \times_S Y}(\mathscr{P}_a \boxtimes_S \mathscr{P}'_b, \mathscr{G}_X \boxtimes_S \mathscr{G}_Y) \\
&= \operatorname{RHom}_{X \times_S Y}(\mathscr{F}_X \boxtimes_S \mathscr{F}_Y, \mathscr{G}_X \boxtimes_S \mathscr{G}_Y) \qquad \square
\end{aligned}$$

A.2.2 Shriek preliminaries

Lemma A.2.2.1. *Suppose X is a ($\star_{\mathrm{F}}$) derived stack over $S = \operatorname{Spec} k$, $\mathscr{F}, \mathscr{G} \in \operatorname{DCoh}(X)$. Then, there are natural equivalences*

(i) $\mathbb{D}(\mathscr{F} \boxtimes \mathscr{G}) = \mathbb{D}(\mathscr{F}) \boxtimes \mathbb{D}(\mathscr{G})$.

(ii) $\omega_X \overset{!}{\otimes} \mathscr{G} = \mathscr{G}$

Proof. Note that $(p_2)^!\mathscr{G} = \omega_X \boxtimes \mathscr{G}$. In particular, $\omega_{X^2} = (p_2)^!\omega_X = \omega_X \boxtimes \omega_X$. Part (i) now follows from Prop. A.2.1.1 and the formula $\mathbb{D}(-) = \mathcal{RH}om^{\otimes_X}(-, \omega)$. Part (ii) follows by noting that

$$\omega_X \overset{!}{\otimes} \mathscr{G} = \Delta^!(\omega_X \boxtimes \mathscr{G}) = \Delta^!(p_2)^!\mathscr{G} = \mathscr{G} \qquad \square$$

Lemma A.2.2.2. *Suppose X is a $(\star_{\mathrm{F}})$ derived stack over $S = \operatorname{Spec} k$, and $\mathscr{F}, \mathscr{G} \in \mathrm{DCoh}(X)$. Then, there is a natural equivalence in $\mathrm{QC}(X)$*

$$\mathscr{F} \overset{!}{\otimes} \mathscr{G} \simeq \mathcal{RH}om_X^{\otimes_X}(\mathbb{D}\mathscr{F}, \mathscr{G})$$

Proof. Since Δ is finite, we have a relative adjunction $(\Delta_*, \Delta^!)$ and we may rewrite

$$\begin{aligned} \mathscr{F} \overset{!}{\otimes} \mathscr{G} &= \Delta^!(\mathscr{F} \boxtimes \mathscr{G}) \\ &= (p_1)_*\Delta_*\mathcal{RH}om_X^{\otimes}\left(\mathcal{O}_X, \Delta^!(\mathscr{F} \boxtimes \mathscr{G})\right) \\ &= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\Delta_*\mathcal{O}_X, \mathscr{F} \boxtimes \mathscr{G}\right) \end{aligned}$$

and since $\Delta_*\mathcal{O}_X$ has coherent homotopy sheaves we may apply coherent duality to rewrite this as

$$= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\mathbb{D}(\mathscr{F} \boxtimes \mathscr{G}), \mathbb{D}\Delta_*\mathcal{O}_X\right)$$

Applying Lemma A.2.2.1(i)

$$\begin{aligned} &= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\mathbb{D}(\mathscr{F}) \boxtimes \mathbb{D}(\mathscr{G}), \mathbb{D}\Delta_*\mathcal{O}_X\right) \\ &= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\mathbb{D}(\mathscr{F}) \boxtimes \mathcal{O}_X, \mathcal{RH}om_{X^2}^{\otimes}\left(\mathcal{O}_X \boxtimes \mathbb{D}(\mathscr{G}), \mathbb{D}\Delta_*\mathcal{O}_X\right)\right) \end{aligned}$$

Undoing the above operations on the inner-$\mathcal{RH}om^{\otimes}$:

$$\begin{aligned} &= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\mathbb{D}(\mathscr{F}) \boxtimes \mathcal{O}_X, \mathcal{RH}om_{X^2}^{\otimes}\left(\Delta_*\mathcal{O}_X, \mathbb{D}(\mathcal{O}_X \boxtimes \mathbb{D}(\mathscr{G}))\right)\right) \\ &= (p_1)_*\mathcal{RH}om_{X^2}^{\otimes}\left(\mathbb{D}(\mathscr{F}) \boxtimes \mathcal{O}_X, \Delta_*\left(\omega_X \overset{!}{\otimes} \mathscr{G}\right)\right) \end{aligned}$$

Applying the relative (Δ^*, Δ_*) adjunction

$$= (p_1)_*\Delta_*\mathcal{RH}om_X^{\otimes}\left(\mathbb{D}(\mathscr{F}) \otimes \mathcal{O}_X, \omega_X \overset{!}{\otimes} \mathscr{G}\right)$$

Finally we complete by Lemma A.2.2.1(ii)

$$= \mathcal{RH}om_X^{\otimes}\left(\mathbb{D}(\mathscr{F}), \mathscr{G}\right) \qquad \square$$

Remark A.2.2.3. Lemma A.2.2.2 admits the following reformulation: Define

$$\mathrm{ev}\colon \mathrm{QC}^!(X) \otimes \mathrm{QC}^!(X) \to k\text{-mod} \qquad \mathscr{F} \otimes \mathscr{G} \mapsto \mathrm{R}\Gamma(\mathscr{F} \overset{!}{\otimes} \mathscr{G}) = \mathrm{RHom}_{X^2}^{\otimes k}(\Delta_*\mathcal{O}_X, \mathscr{F} \boxtimes \mathscr{G})$$

Then, the functor $\mathbb{D}(-)\colon \mathrm{DCoh}(X) \to \mathrm{QC}^!(X)^\vee = \mathrm{Fun}^{ex}(\mathrm{DCoh}(X), k\text{-mod})$ is characterized by

$$\mathrm{RHom}^{\otimes k}_X(\mathbb{D}(\mathscr{F}), -) = \mathrm{ev}(\mathscr{F} \otimes -) = \mathrm{R}\Gamma(\mathscr{F} \overset{!}{\otimes} -) = \mathrm{RHom}^{\otimes k}_{X^2}(\Delta_* \mathcal{O}_X, \mathscr{F} \boxtimes -)$$

Grothendieck duality implies that this is part of a duality datum giving $\mathrm{QC}^!(X) \simeq \mathrm{QC}^!(X)^\vee$.

Theorem A.2.2.4. *Suppose $S = \operatorname{Spec} k$ is a perfect field; that X, Y are almost finitely-presented ($\star_{\mathrm{F}}$) stacks over S; and that $Z_X \subset X$, $Z_Y \subset Y$ are closed subsets. Then, there are equivalences of categories*

$$\begin{array}{ccc} \mathrm{Fun}^L_k(\mathrm{QC}^!_{Z_X}(X), \mathrm{QC}^!_{Z_Y}(Y)) & \xleftarrow[\sim]{\Phi^!} & \mathrm{QC}^!_{Z_X \times_S Z_Y}(X \times_S Y) \\ \Psi^! \uparrow \sim & \overset{\sim}{\underset{\boxtimes}{\nearrow}} & \\ \mathrm{QC}^!_{Z_X}(X) \widehat{\otimes}_k \mathrm{QC}^!_{Z_Y}(Y) & & \end{array}$$

where

- *$\boxtimes$ denotes external tensor product over S, and restricts to an equivalence on compact objects*
$$\boxtimes\colon \mathrm{DCoh}_{Z_X}(X) \otimes_k \mathrm{DCoh}_{Z_Y}(Y) \xrightarrow{\sim} \mathrm{DCoh}_{Z_X \times_S Z_Y}(X \times_S Y)$$

- $\Phi^!(\mathscr{K}) = (p_2)_* \left(p_1^!(-) \overset{!}{\otimes} \mathscr{K} \right)$ *is the !-Fourier-Mukai functor with kernel $\mathscr{K}$.*

- $\Psi^!(\mathscr{F} \otimes \mathscr{G}) = \mathrm{Hom}_{X/S}(\mathbb{D}(\mathscr{F}), -) \otimes_k \mathscr{G}$ *for $\mathscr{F}, \mathscr{G}$ compact objects.*

Restricting to the case $X = Y$:

- $\mathrm{id}_{\mathrm{QC}^!_Z(X)} = \Phi^!(\omega_{\Delta,Z})$*, where $\omega_{\Delta,Z} = \Delta_* \mathrm{R}\Gamma_Z(\omega_X)$ and $\omega_X = \mathbb{D}(\mathcal{O}_X)$ is the dualizing complex and $\mathrm{R}\Gamma_Z(-)\colon \mathrm{QC}^!(X) \to \mathrm{QC}^!_Z(X)$ is (the Ind-coherent version of) local cohomology along Z.*

- *More generally,* $\Phi^!(\Delta_* \mathscr{F}) = \mathscr{F} \overset{!}{\otimes} -$.

- $\mathrm{ev}(\Phi^!(\mathscr{K})) = \mathrm{Hom}_{\mathrm{QC}^!(X^2)}(\Delta_* \mathcal{O}_X, \mathscr{K})$ *(no support condition!).*

Proof. The Grothendieck Duality anti-equivalence respects supports and so restricts to $\mathbb{D}(-)\colon \mathrm{DCoh}_{Z_X}(X)^{\mathrm{op}} \simeq \mathrm{DCoh}_{Z_X}(X)$. This implies that $\mathrm{QC}^!_{Z_X}(X)$ is self-dual over $\mathrm{QC}(S)$ via $\mathbb{D}(-)$, so that $\Psi^!$ is an equivalence (it does not even matter that the target category is of geometric origin). We will now verify commutativity of the diagram, the indicated formulas, and only finally that the relevant maps are equivalences.

Diagram commutes:
Let us prove that the diagram commutes up to natural equivalence. Since each of $\boxtimes$, $\Psi^!$, and $\Phi^!$ is colimit preserving it suffices to give a natural equivalence $\Psi^!_{\mathscr{F} \otimes \mathscr{G}} = \Phi^!_{\mathscr{F} \boxtimes \mathscr{G}}$ for $\mathscr{F} \in \mathrm{DCoh}(X)$, $\mathscr{G} \in \mathrm{DCoh}(Y)$. Since both functors are colimit preserving, we may check

this for $T \in \mathrm{DCoh}(X)$:

$$\begin{aligned}
\Phi^!_{\mathscr{F}\boxtimes\mathscr{G}}(T) &= (p_2)_*\left((p_1)^! T \overset{!}{\otimes} (\mathscr{F}\boxtimes\mathscr{G})\right)\\
&= (p_2)_*\left((T\boxtimes\omega_Y) \overset{!}{\otimes} (\mathscr{F}\boxtimes\mathscr{G})\right)\\
&= (p_2)_* \operatorname{RHom}^{\otimes}_{X\times Y}\left(\mathbb{D}(\mathscr{F})\boxtimes\mathbb{D}(\mathscr{G}), T\boxtimes\omega_Y\right)\\
&= (p_2)_*\left(\operatorname{RHom}^{\otimes}_X(\mathbb{D}(\mathscr{F}),T)\boxtimes\operatorname{RHom}^{\otimes}_Y(\mathbb{D}(\mathscr{G}),\omega_Y)\right)\\
&= (p_2)_*\left(\operatorname{RHom}^{\otimes}_X(\mathbb{D}(\mathscr{F}),T)\boxtimes\operatorname{RHom}^{\otimes}_Y(\mathcal{O}_Y,\mathscr{G})\right)\\
&= \mathrm{R}\Gamma\left(\operatorname{RHom}^{\otimes}_X(\mathbb{D}(\mathscr{F}),T)\right)\otimes_k\mathscr{G}\\
&= \operatorname{Map}_X(\mathbb{D}(\mathscr{F}),T)\otimes_k\mathscr{G}\\
&= \Psi^!_{\mathscr{F}\otimes\mathscr{G}}(T)
\end{aligned}$$

Here have have implicitly used Lemma A.2.2.1, Lemma A.2.2.2, and coherent duality.

Formulaire: We first prove that $\Phi^!_{\Delta_*\mathscr{F}}(-) = - \overset{!}{\otimes}\mathscr{F}$. Extending by colimits, it suffices to note that for $T, \mathscr{F}\in\mathrm{DCoh}(X)$

$$\begin{aligned}
\Phi^!_{\Delta_*\mathscr{F}}(T) &= (p_2)_*\left((p_1)^! T\overset{!}{\otimes}\Delta_*\mathscr{F}\right)\\
&= (p_2)_*\operatorname{RHom}^{\otimes}_{X^2}\left(\mathbb{D}(T)\boxtimes\mathcal{O}_Y,\Delta_*\mathscr{F}\right)\\
&= (p_2)_*\Delta_*\operatorname{RHom}^{\otimes}_X\left(\Delta^*(\mathbb{D}(T)\boxtimes\mathcal{O}_Y),\mathscr{F}\right)\\
&= \operatorname{RHom}^{\otimes}_X(\mathbb{D}(T),\mathscr{F})\\
&= T\overset{!}{\otimes}\mathscr{F}
\end{aligned}$$

By Lemma A.2.2.1(ii), it follows that $\Phi^!_{\Delta_*\omega_X} = \mathrm{id}_{\mathrm{QC}^!(X)}$. More generally, setting $\omega_{\Delta,Z} = \Delta_*\mathrm{R}\Gamma_Z(\omega_X)$, we see that

$$\Phi^!_{\omega_{\Delta,Z}}(T) = T\overset{!}{\otimes}\mathrm{R}\Gamma_Z(\omega_X) = \mathrm{R}\Gamma_Z(T)\overset{!}{\otimes}\omega_X = \mathrm{R}\Gamma_Z(T).$$

Since $\mathrm{R}\Gamma_Z$ is the identify functor on $\mathrm{QC}^!_Z(X)$, we obtain $\Phi^!_{\omega_{\Delta,Z}} = \mathrm{id}_{\mathrm{QC}^!_Z(X)}$ in case of supports.

To check the formula for the trace, it suffices (since both sides preserve colimits in both variables) to check it in case $\mathscr{K} = \mathscr{F}\boxtimes\mathscr{G}$ with $\mathscr{F},\mathscr{G}\in\mathrm{DCoh}(X)$. Applying Lemma A.2.2.2 we see that

$$\operatorname{ev}\left(\Phi^!_{\mathscr{K}}\right) = \operatorname{ev}\left(\Psi^!_{\mathscr{F}\otimes\mathscr{G}}\right) = \operatorname{Map}_X\left(\mathbb{D}(\mathscr{F}),\mathscr{G}\right) = \mathrm{R}\Gamma\left(\mathscr{F}\overset{!}{\otimes}\mathscr{G}\right) = \operatorname{Map}_{\mathrm{QC}^!(X^2)}\left(\Delta_*\mathcal{O}_X,\mathscr{F}\boxtimes\mathscr{G}\right)$$

Equivalences:
Since the diagram commutes and $\Psi^!$ is an equivalence, it suffices to show that $\boxtimes$ is an equivalence. By Prop. A.2.1.1 it preserves compact objects and is fully faithful. It suffices to show that it is essentially surjective on compact objects. In Prop. A.2.3.2 below, we we

handle the case without support conditions. Let us show how this implies the general case:

$$\begin{array}{ccc} \mathrm{DCoh}(Z_X)\otimes \mathrm{DCoh}(Z_Y) & \xrightarrow{\sim} & \mathrm{DCoh}(Z_X\times Z_Y) \\ \downarrow & & \downarrow \\ \mathrm{DCoh}_{Z_X}(X)\otimes \mathrm{DCoh}_{Z_Y}(Y) & \longrightarrow & \mathrm{DCoh}_{Z_X\times Z_Y}(X\times Y) \end{array}$$

We have seen that the bottom horizontal arrow is fully faithful, so since both categories are stable and idempotent complete it suffices to show that it has dense image. We have seen that the the top horizontal arrow is an equivalence. The right vertical arrow has dense image by Lemma 2.2.0.2. Consequently, the bottom horizontal arrow has dense image as desired. □

A.2.3 Devissage

Lemma A.2.3.1. *Suppose X is a quasi-compact quasi-separated scheme and $U \subset X$ is a quasi-compact open, with closed complement $Z = X - U$. Suppose that $\boxtimes$ is an equivalence for the pairs (U, Y) and (Z, Y). Then, it is an equivalence for (X, Y).*

Proof. Observe that $\mathrm{DCoh}(Z\times_S Y) \to \mathrm{DCoh}_{Z\times_k Y}(X\times_S Y)$ has dense image by Lemma 2.2.0.2 and filtering by powers of the ideal sheaf of Z. Considering the diagram

$$\begin{array}{ccc} \mathrm{DCoh}(Z)\otimes_k \mathrm{DCoh}(Y) & \longrightarrow & \mathrm{DCoh}(Z\times_S Y) \\ \downarrow & & \downarrow \\ \mathrm{DCoh}_Z(X)\otimes_k \mathrm{DCoh}(Y) & \longrightarrow & \mathrm{DCoh}_{Z\times_k Y}(X\times_S Y) \end{array}$$

we see that the right vertical arrow has dense image; since the top horizontal arrow does by assumption, so does the bottom horizontal arrow.

Consider the diagram

$$\begin{array}{ccccc} \mathrm{DCoh}_Z(X)\otimes_k \mathrm{DCoh}(Y) & \longrightarrow & \mathrm{DCoh}(X)\otimes_k \mathrm{DCoh}(Y) & \longrightarrow & \mathrm{DCoh}(U)\otimes_k \mathrm{DCoh}(Y) \\ \downarrow & & \downarrow & & \downarrow\sim \\ \mathrm{DCoh}_{Z\times_S Y}(X\times_S Y) & \longrightarrow & \mathrm{DCoh}(X\times_S Y) & \longrightarrow & \mathrm{DCoh}(U\times_S Y) \end{array}$$

We claim both rows are Verdier-Drinfeld sequences. For the bottom row, this is the usual localization sequence of a closed subset for DCoh. For the top row, reduce to the usual localization sequence by Lemma 3.1.4.2. Set $\mathcal{A} = \langle \mathrm{im}\,\boxtimes_{X,Y}\rangle \subset \mathrm{DCoh}(X\times_S Y)$. We will show that $\mathcal{A} = \mathrm{DCoh}(X\times_S Y)$, using the following categorical version of the "5-lemma":

Examining the left-most arrow, we see that $\mathcal{A}$ contains $\mathrm{DCoh}_{Z\times_S Y}(X\times_S Y)$. Letting $\overline{\mathcal{A}}$ denote its image in the Verdier quotient $\mathrm{DCoh}(U\times_S Y)$, it suffices to show that $\overline{\mathcal{A}}$ is dense in $\mathrm{DCoh}(U\times_S Y)$. Since the right-most vertical arrow is an equivalence, this follows from observing that $\mathrm{DCoh}(X)\otimes_k \mathrm{DCoh}(Y) \to \mathrm{DCoh}(U)\otimes_k \mathrm{DCoh}(Y)$ has dense image. □

Proposition A.2.3.2. *Suppose k is a perfect field, $S = \operatorname{Spec} k$, and that X, Y are almost finitely-presented ($\star_{\mathrm{F}}$) derived stack over S. Then, the exterior product induces equivalences*

$$\boxtimes\colon \mathrm{DCoh}(X)\otimes_k \mathrm{DCoh}(Y) \xrightarrow{\sim} \mathrm{DCoh}(X\times Y)$$

$$\boxtimes\colon \operatorname{Ind}\operatorname{DCoh}(X)\widehat{\otimes}_k \operatorname{Ind}\operatorname{DCoh}(Y) \xrightarrow{\sim} \operatorname{Ind}\operatorname{DCoh}(X\times Y)$$

This remains true with support conditions.

Proof. We have seen how to reduce the case with support conditions to that without in Theorem A.2.2.4. Also, note that it suffices to prove either the small or the Ind-completed version.

Suppose $U_\bullet \to X$ is an étale cover, so that $\operatorname{Ind}\operatorname{DCoh} X = \operatorname{Tot}\{\operatorname{Ind}\operatorname{DCoh} U_\bullet\}$ by Theorem A.1.2.5; since $U_\bullet \times Y \to X\times Y$ is again an étale cover, we also have $\operatorname{Ind}\operatorname{DCoh}(X\times Y) = \operatorname{Tot}\{\operatorname{Ind}\operatorname{DCoh} U_\bullet \times U\}$. Since $\operatorname{Ind}\operatorname{DCoh}(Y)$ is dualizable over k-mod by Lemma 3.2.2.1, $-\widehat{\otimes}_k \operatorname{Ind}\operatorname{DCoh}(Y)$ preserves arbitrary limits. Consequently, we have a diagram of equivalences

$$\begin{aligned}\operatorname{Ind}\operatorname{DCoh}(X)\widehat{\otimes}_k \operatorname{Ind}\operatorname{DCoh}(Y) &\xrightarrow{\pi^*\otimes \mathrm{id}} \operatorname{Tot}\{\operatorname{Ind}\operatorname{DCoh}(U_\bullet)\}\,\widehat{\otimes}_k \operatorname{Ind}\operatorname{DCoh}(Y)\\ &\xrightarrow{\sim} \operatorname{Tot}\{\operatorname{Ind}\operatorname{DCoh}(U_\bullet\times Y)\} \xleftarrow{(\pi,\mathrm{id})^*} \operatorname{Ind}\operatorname{DCoh}(X\times Y)\end{aligned}$$

Exterior product commutes with finite Tor-dimension pullbacks, so we conclude that our claim is local on X. Similarly, it is local on Y. Consequently, we may reduce to the case of X and Y affine derived schemes.

We will now prove the small, idempotent complete, variant. Since $\boxtimes$ is fully faithful by Prop. A.2.1.1, it suffices to prove that it is essentially surjective. Since both the tensor product and $\operatorname{DCoh}(X\times_S Y)$ are stable and idempotent complete, it suffices to show that the image of $\boxtimes$ is *dense* in the sense that its thick-closure is the whole category.

Step 1. Case of X, Y regular (discrete) schemes:
The analogous statement is well-known (see, e.g., Töen or [BZFN]) with DCoh replaced by Perf throughout. Since X, Y are regular we have $\operatorname{DCoh}(X) = \operatorname{Perf}(X)$, $\operatorname{DCoh}(Y) = \operatorname{Perf}(Y)$. It remains to observe that $X\times_S Y$ is again regular, since X, Y are finite-type over a perfect field k. Consequently, $\operatorname{DCoh}(X\times_S Y) = \operatorname{Perf}(X\times_S Y)$, and we're done.

Step 2. Reduction to the case X, Y reduced (discrete) schemes:
Consider the natural map $i\colon (\pi_0 X)_{\mathrm{red}} \to X$. Under our finiteness hypotheses, it is proper and consequently we obtain a functor $i_*\colon \operatorname{DCoh}((\pi_0X)_{\mathrm{red}}) \to \operatorname{DCoh}(X)$. The standard filtration argument shows that every object of $\operatorname{DCoh}(X)$ admits a filtration with associated graded in the image of i_*, from which it follows that $i_*\colon \operatorname{DCoh}((\pi_0X)_{\mathrm{red}}) \to \operatorname{DCoh}(X)$ has dense image.

Consider the diagram

$$\begin{array}{ccc}\operatorname{DCoh}(X)\otimes_k \operatorname{DCoh}(Y) & \xrightarrow{\boxtimes} & \operatorname{DCoh}(X\times_S Y)\\ \uparrow & & \uparrow\\ \operatorname{DCoh}((\pi_0X)_{\mathrm{red}})\otimes_k \operatorname{DCoh}((\pi_0Y)_{\mathrm{red}}) & \xrightarrow[\boxtimes_{\mathrm{red}}]{} & \operatorname{DCoh}((\pi_0X)_{\mathrm{red}}\times_S(\pi_0Y)_{\mathrm{red}})\end{array}$$

If $\boxtimes_{\mathrm{red}}$ has dense image then so does $\boxtimes$, since the right vertical arrow has dense image by the above (the map $[\pi_0(X\times_S Y)]_{\mathrm{red}} \to X\times_S Y$ factors through $(\pi_0X)_{\mathrm{red}}\times_S(\pi_0Y)_{\mathrm{red}}$, and is in fact an equivalence under our hypotheses).

Step 3. Reduction to the case X, Y integral (discrete) schemes:
By Step 2, we may assume X, Y are reduced schemes. Since they are finite-type over a field, they have finitely-many irreducible components $X_1,\ldots,X_n$, $Y_1,\ldots,Y_m$. Using Lemma A.2.3.1, we may induct on the number of irreducible components.

Step 4. Completing the proof:
By the above, we may suppose X, Y are integral schemes. By Noetherian induction, we may suppose the claim is known for all pairs (X', Y') such that $\dim X' \leq \dim X$, $\dim Y' \leq \dim Y$ with at least one of these inequalities is strict. Since X, Y are integral and of finite-type over a perfect field, they are generically regular. Let $U \subset X$, $V \subset Y$ be dense open regular subsets, and $Z_X = X - U$, $Z_Y = Y - V$. Using Lemma A.2.3.1, we see that the claim holds for (X, Y) if it holds for (U, V), (Z_X, V), (U, Z_Y), and (Z_X, Z_Y): The first of these follows by Step 1, while the rest follow by the inductive hypothesis. □

Remark A.2.3.3. After reducing to the case of a reduced discrete scheme, one can also conclude quite quickly using de Jong's alterations and Lemma A.1.2.4(ii): Using de Jong's alterations one may produce proper surjective maps $p \colon \widetilde{X} \to X$, $q \colon \widetilde{Y} \to Y$ with $\widetilde{X}$ and $\widetilde{Y}$ regular. Then, Lemma A.1.2.4(ii) identifies $\mathrm{QC}^!(X) = (p^! p_*)\text{-mod}\, \mathrm{QC}^!(\widetilde{X})$ and similarly for $\mathrm{QC}^!(Y)$ (using q) and for $\mathrm{QC}^!(X \times Y)$ (using $p \times q$). Since $\widetilde{X}$ and $\widetilde{Y}$ are regular, as is their product, we know that $\mathrm{QC}^!(\widetilde{X}) \widehat{\otimes}_k \mathrm{QC}^!(\widetilde{Y}) = \mathrm{QC}^!(\widetilde{X} \times \widetilde{Y})$. Finally, it suffices to identify $(p \times q)^! (p \times q)_*$ with the algebraic tensor-product monad.

A.2.4 Extensions

Proposition A.2.4.1. *Suppose S is regular ($\star$) stack.*

(i) Suppose $Y \to S$ is a smooth relative scheme. Then,

$$\boxtimes \colon \mathrm{DCoh}(X) \otimes_S \mathrm{DCoh}(Y) \longrightarrow \mathrm{DCoh}(X \times_S Y)$$

is an equivalence for all excellent (i.e., $\pi_0 X$ is an excellent ordinary scheme) derived stacks X over S. If S is excellent (in the sense that all schemes of finite-type over it are excellent), then this holds for any almost finitely-presented ($\star$) derived stack over S.

(ii) Suppose S is regular and excellent; that X, Y are ($\star$) derived DM stacks over S; and that $Z_X \subset X$, $Z_Y \subset Y$ are closed subsets. Suppose furthermore that Z_Y, with its reduced induced scheme structure, is smooth over S. Then,

$$\boxtimes \colon \mathrm{DCoh}_{Z_X}(X) \otimes_S \mathrm{DCoh}_{Z_Y}(Y) \longrightarrow \mathrm{DCoh}_{Z_X \times_S Z_Y}(X \times_S Y)$$

is an equivalence.

Proof.

(i) The second sentence follows from the first. As in Prop. A.2.3.2, the question is local on X so that we may suppose X is affine. If X is regular, than so is $X \times_S Y$ (being smooth over X) and we are done by the analogous statement for Perf. Otherwise, we may proceed by Noetherian induction on X, as in the proof of Prop. A.2.3.2. Since all derived schemes occurring in the Noetherian induction will be almost finitely-presented over X, they will all be Noetherian and excellent. As there, we reduce to the case of X discrete and reduced, and apply Lemma A.2.3.1 to reduce to the case of X integral. By excellence, there is an open dense subset on which X is regular and applying Lemma A.2.3.1 the Noetherian induction continues.

(ii) We will reduce to the case of Y smooth over S and without support conditions i.e., (i): As before, this is local on X and Y, so that we may suppose they are affine. Let Z_X, Z_Y denote the reduced induced scheme structures on the closed subsets, and consider the diagram

$$\begin{array}{ccc} \mathrm{DCoh}(Z_X) \otimes_S \mathrm{DCoh}(Z_Y) & \longrightarrow & \mathrm{DCoh}(Z_X \times_S Z_Y) \\ \downarrow & & \downarrow \\ \mathrm{DCoh}_{Z_X}(X) \otimes_S \mathrm{DCoh}_{Z_Y}(Y) & \longrightarrow & \mathrm{DCoh}_{Z_X \times_S Z_Y}(X \times Y) \end{array}$$

The horizontal maps are fully faithful by Prop. A.2.1.1, so it suffices to prove that the bottom horizontal map has dense image; but, the right-hand vertical arrow has dense image. This reduces us to showing that the top horizontal map is an equivalence. □

A.2.5 Hochschild-type invariants of coherent complexes

Corollary A.2.5.1. *Suppose X is a finite-type ($\star_{\mathrm{F}}$) derived stack over a perfect field k. Then, Grothendieck duality induces*

(i) An isomorphism

$$\mathbf{HH}^\bullet(\mathrm{DCoh}(X)) \xrightarrow{\sim} \mathbf{HH}^\bullet(\mathrm{Perf}(X))$$

of Hochschild cochain complexes.

(ii) A "Poincaré duality"

$$\mathbf{HH}_\bullet(\mathrm{DCoh}(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{QC}(X)}(\Delta^*\Delta_*\mathcal{O}_X, \omega_X) = \mathrm{R\Gamma}\left[X, \mathbb{D}\left(\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(X))\right)\right]$$

(iii) Suppose $Z \subset X$ a closed subset. Then,

$$\mathbf{HH}_\bullet(\mathrm{DCoh}_Z(X)) \xrightarrow{\sim} \mathrm{R\Gamma}_Z\left(\underline{\mathbf{HH}}_\bullet(\mathrm{DCoh}(X))\right)$$

Proof.

(i) Recall that $\mathrm{id}_{\mathrm{QC}(X)} = \Phi_{\mathcal{O}_\Delta}$ and $\mathrm{id}_{\mathrm{QC}^!(X)} = \Phi^!_{\omega_\Delta}$ (Theorem A.2.2.4). So,

$$\begin{aligned} \mathbf{HH}^\bullet(\mathrm{DCoh}\, X) &= \mathrm{Hom}_{\mathrm{Fun}^L(\mathrm{QC}^!(X), \mathrm{QC}^!(X))}(\mathrm{id}, \mathrm{id}) \\ &= \mathrm{Hom}_{\mathrm{QC}^!(X^2)}(\omega_\Delta, \omega_\Delta) \\ &= \mathrm{Hom}_{\mathrm{DCoh}(X^2)}(\mathbb{D}\mathcal{O}_\Delta, \mathbb{D}\mathcal{O}_\Delta) \\ &= \mathrm{Hom}_{\mathrm{DCoh}(X^2)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \\ &= \mathrm{Hom}_{\mathrm{QC}(X^2)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \\ &= \mathrm{Hom}_{\mathrm{Fun}^L(\mathrm{QC}(X), \mathrm{QC}(X))}(\mathrm{id}, \mathrm{id}) \\ &= \mathbf{HH}^\bullet(\mathrm{Perf}\, X) \end{aligned}$$

(ii) Recall that $\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(X)) = \Delta^*\Delta_*\mathcal{O}_X$ is the sheafified Hochschild homology of Perf X.

Then, Theorem A.2.2.4 implies

$$\begin{aligned}
\mathbf{HH}_\bullet(\mathrm{DCoh}\,X) &= \mathrm{ev}(\mathrm{id}_{\mathrm{QC}^!(X)}) \\
&= \mathrm{Hom}_{\mathrm{QC}^!(X^2)}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X) \\
&= \mathrm{Hom}_{\mathrm{DCoh}(X^2)}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X) \\
&= \mathrm{Hom}_{\mathrm{QC}(X^2)}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X) \\
&= \mathrm{Hom}_{\mathrm{QC}(X)}(\Delta^*\Delta_*\mathcal{O}_X, \omega_X) \\
&= \mathrm{R}\Gamma\left(\mathbb{D}(\Delta^*\Delta_*\mathcal{O}_X)\right)
\end{aligned}$$

(iii) Recall that

$$\underline{\mathbf{HH}}_\bullet(\mathrm{DCoh}\,X) = (p_1)_*\mathcal{RH}om_{\mathrm{QC}^!(X^2)}\left(\mathcal{O}_X, \Delta^!\Delta_*\omega_X\right) = \Delta^!\Delta_*\omega_X$$

and that

$$\begin{aligned}
\mathbf{HH}_\bullet(\mathrm{DCoh}_Z\,X) &= \mathrm{Map}_{\mathrm{QC}^!(X^2)}\left(\Delta_*\mathcal{O}_X, \Delta_*\underline{\mathrm{R}\Gamma}_Z\omega_X\right) \\
&= \mathrm{Map}_{\mathrm{QC}^!(X)}\left(\mathcal{O}_X, \Delta^!\Delta_*\underline{\mathrm{R}\Gamma}_Z\omega_X\right) \\
&= \mathrm{R}\Gamma\left(\Delta^!\Delta_*\underline{\mathrm{R}\Gamma}_{Z^2}\omega_X\right)
\end{aligned}$$

Note that $\Delta^! \circ \underline{\mathrm{R}\Gamma}_{Z^2} \simeq \mathrm{R}\Gamma_Z \circ \Delta^!$ (the left adjoints coincide), and that the natural map $\Delta_* \circ \underline{\mathrm{R}\Gamma}_Z \xrightarrow{\sim} \underline{\mathrm{R}\Gamma}_{Z^2} \circ \Delta_*$ is an equivalence (e.g., using the Cech-nerve description of Section 4.1). So, we conclude

$$\begin{aligned}
&= \mathrm{R}\Gamma\left(\Delta^!\underline{\mathrm{R}\Gamma}_{Z^2}\Delta_*\omega_X\right) \\
&= \mathrm{R}\Gamma_Z\left(\Delta^!\Delta_*\omega_X\right)
\end{aligned}$$

□

Remark A.2.5.2. In particular, item (ii) implies that

$$U \mapsto \mathbf{HH}_\bullet(\mathrm{DCoh}(U))$$

forms a sheaf of quasi-coherent complexes, which we'll denote $\underline{\mathbf{HH}}_\bullet(\mathrm{DCoh}\,X)$. (This can also be seen directly.) Then, (ii) may be reformulated as the (more evidently a duality) assertion that

$$\underline{\mathbf{HH}}_\bullet(\mathrm{DCoh}(X)) = \mathbb{D}\left(\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(X))\right)$$

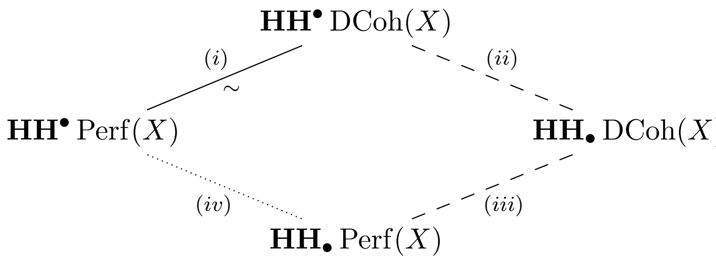
If X is proper, this implies a (vector space) duality on global sections.

Note that this really is using duality: In the case that X is smooth over a characteristic zero field, and identifying $\underline{\mathbf{HH}}_\bullet(\mathrm{Perf}(X)) = \Omega^\bullet_X$ via HKR, this is a reflection of the sheaf perfect-pairing $\wedge\colon \Omega^\bullet_X \otimes \Omega^\bullet_X \to \omega_X$ (where $\Omega^\bullet_X = \oplus_i \Omega_X[i]$).

Remark A.2.5.3. Meanwhile, item (i) seems somewhat bizarre. It does, however, lead to the following observation:

Suppose X is lci over a perfect field, and that (for simplicity) X is affine. Then, it is (?) known that thick subcategories of $\mathrm{DCoh}(X)$ may be classified by $\mathbb{G}_m$-equivariant specialization-closed subsets of $\operatorname{Spec} \pi_* \mathbf{HH}^\bullet(\operatorname{Perf} X)$. Using the above, we may interpret this latter space as intrinsic to $\operatorname{DCoh}(X)$.

Remark A.2.5.4. Cor. A.2.5.1 may be flushed out to the following picture:



(i) Are *isomorphic* by the Corollary.

(ii) Differ by a *shift* provided X is Calabi-Yau in the very weak sense that $\omega_X \simeq \mathcal{O}_X[-d]$ for some d. (For this, X need not be smooth. For instance, any Gorenstein local ring is Calabi-Yau in this sense.) Indeed, Theorem A.2.2.4 allows us to identify

$$\mathbf{HH}^\bullet \operatorname{DCoh}(X) = \operatorname{Hom}_{\operatorname{QC}^!(X^2)}(\omega_\Delta, \omega_\Delta) = \operatorname{Hom}_{\operatorname{QC}(X)}(\Delta^* \Delta_* \mathcal{O}_X, \mathcal{O}_X)$$

$$\mathbf{HH}_\bullet \operatorname{DCoh}(X) = \operatorname{Hom}_{\operatorname{QC}^!(X^2)}(\mathcal{O}_\Delta, \omega_\Delta) = \operatorname{Hom}_{\operatorname{QC}(X)}(\Delta^* \Delta_* \mathcal{O}_X, \omega_X)$$

(iii) Are *linearly dual* provided that X is proper. A sheafified ("local") version of this duality holds always, by Remark A.2.5.2.

(iv) Are *dual up to a shift* provided X is proper and Calabi-Yau. (This is very well-known, at least when X is also regular.)

A.2.6 Case of hypersufaces

A.2.6.1. In this subsection (M, f) is an LG pair, $M_0 = M \times_{\mathbb{A}^1} 0$ is the derived fiber product, and $i \colon M_0 \to M$ is the inclusion. The goal of this subsection is to make explicit the Hochschild invariants of the terms of the Verdier-Drinfeld sequence

$$\operatorname{Perf}(M_0) \to \operatorname{DCoh}(M_0) \to \operatorname{DSing}(M_0)$$

for the purposes of comparison to Theorem 6.1.2.5.

Proposition A.2.6.2. *With notation as above, and supposing that M is a scheme in the HKR-type statements, we have natural isomorphisms*

$$\underline{\mathbf{HH}}_\bullet^{/k}(\operatorname{Perf}(M_0)) = i^*\left(\underline{\mathbf{HH}}_\bullet(\operatorname{Perf}(M))_{B\widehat{\mathbb{G}_a}}\right) \simeq i^*\left(\Omega_M^\bullet[u^\ell/\ell!], du = df\right)$$

$$\underline{\mathbf{HH}}_\bullet^{/k}(\operatorname{DCoh}(M_0)) = i^!\left(\underline{\mathbf{HH}}_\bullet(\operatorname{DCoh}(M))^{B\widehat{\mathbb{G}_a}}\right) \simeq i^!\left(\Omega_M^\bullet[\![\beta]\!], -\beta \cdot df \wedge\right)$$

$$\underline{\mathbf{HH}}_\bullet^{/k}(\operatorname{DSing}(M_0)) = i^!\left(\underline{\mathbf{HH}}_\bullet(\operatorname{DCoh}(M))^{\mathrm{Tate}}\right) \simeq i^!\left(\Omega_M^\bullet(\!(\beta)\!), -\beta \cdot df \wedge\right)$$

Furthermore, the maps induced by the Verdier-Drinfeld sequence identify with $i^!$ applied to the Tate sequence for $\underline{\mathbf{HH}}_\bullet(\operatorname{DCoh}(M))$ via the identification

$$i^*\left(\underline{\mathbf{HH}}_\bullet(\operatorname{Perf}(M))_{B\widehat{\mathbb{G}_a}}\right) = i^!\left(\underline{\mathbf{HH}}_\bullet(\operatorname{DCoh}(M))_{B\widehat{\mathbb{G}_a}}[+1]\right)$$

coming from noting that $\mathrm{Perf}(M) = \mathrm{DCoh}(M)$ *and* $i^* = i^![+1]$.

Proof. We are implicitly using the $B\widehat{\mathbb{G}_a}$ action on $\mathrm{Perf}(M) = \mathrm{DCoh}(M)$, and the computation of the induced action on $\underline{\mathbf{HH}_\bullet}$, of 6.1.2.1. The computations then follow from the functor theorems for Perf and DCoh, by using base-change in the square

$$\begin{array}{ccc} M_0 & \xrightarrow{\ i\ } & M \\ {\scriptstyle \Delta_{M_0}}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\ (M_0)^2 & \xrightarrow[k]{} & (M^2)_0 \end{array}$$

to rewrite for instance

$$\begin{aligned} \mathbf{HH}_\bullet^{/k}(\mathrm{DCoh}(M_0)) &= \mathrm{RHom}_{(M_0)^2}\left((\Delta_{M_0})_* i^* \mathcal{O}_M, (\Delta_{M_0})_* i^! \omega_M\right) \\ &= \mathrm{RHom}_{(M_0)^2}\left((\Delta_{M_0})_* i^* \mathcal{O}_M, k^! \overline{\Delta}_* \omega_M\right) \\ &= \mathrm{R}\Gamma\left(i^! \overline{\Delta}^! \overline{\Delta}_* \omega_M\right) \end{aligned}$$

and then noting that $\overline{\Delta}^! \overline{\Delta}_* \omega_M = \underline{\mathbf{HH}_\bullet}(\mathrm{DCoh}(M))^{B\widehat{\mathbb{G}_a}}$ by the proof of Theorem 6.1.2.5. □

Remark A.2.6.3. To summarize, the difference between the formulas appearing in the previous Proposition and those in Theorem 6.1.2.5 is the following: Here we take $i^!$, while there we take $\mathrm{R}\Gamma_{M_0}$.

– Recall that $\mathrm{R}\Gamma_{M_0}$ may be regarded as shriek-pullback to the formal completion $\widehat{M_0}$. Consequently for any complex $\mathscr{F}$ on M: $\mathrm{R}\Gamma_{M_0}(\mathscr{F})$ is $k[x]$-linear, and $i^!(\mathscr{F}) = \mathrm{fib}\,(x\colon \mathrm{R}\Gamma_{M_0}(\mathscr{F}) \to \mathrm{R}\Gamma_{M_0}(\mathscr{F}))$.

– The above is a general phenomenon: If $\mathcal{C}$ is a $k[\![\beta]\!]$-linear category, then $\mathbf{HH}_\bullet^{k[\![\beta]\!]}(\mathcal{C})$ is naturally $k[\![\beta]\!] \otimes k[\![x]\!]$-linear and $\mathbf{HH}_\bullet^{k}(\mathcal{C}) = \mathrm{fib}\left(x\colon \mathbf{HH}^{k[\![\beta]\!]}(\mathcal{C}) \to \mathbf{HH}^{k[\![\beta]\!]}(\mathcal{C})\right)$.

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