SOME REMARKS ON SHIFTED SYMPLECTIC STRUCTURES ON NON-COMPACT MAPPING SPACES

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Abstract. We show existence of shifted symplectic structures on some stacks closedly related to mapping stacks from non-proper sources. The motivating example is the moduli stack of stable pairs on noncompact threefolds.

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1. Introduction

The present paper is intended to provide a utility result for [M2]. More precisely, we show the existence of (-1)-shifted symplectic structures on moduli spaces of stable pairs on noncompact CY threefolds. This is Cor. 4.0.9.

To better explain both the goal and the wrinkle in the argument, let us first explain how the arugment might go in the compact CY threefold case. In this case we proceed in steps left-to-right through the following diagram

$$\mathrm{Perf} \qquad \mathrm{Perf}(X) \longleftarrow \mathrm{Perf}^{\mathcal{O}_X[+1]}(X) \hookleftarrow \{\text{stable pairs on } X\}$$

(i) One begins with the argument for mapping stacks in [PTVV]: Since the stack Perf carries a 2-shifted symplectic structure, [PTVV] shows that a universal pullback-and-integrate-along-X construction produces a (-1)-shifted symplectic structure on the Hom-stack $\mathrm{Perf}(X) := \mathrm{Hom}(X, \mathrm{Perf})$ provided that X is a *proper* 3-CY. The non-degenerate pairing on tangent spaces induced by the symplectic form is nothing but the sheafy trace pairing combined with the Grothendieck trace

$$\mathrm{RHom}_X(\mathscr{E},\mathscr{E})^{\otimes 2} \xrightarrow{\mathrm{tr}} \mathrm{R\Gamma}(X,\mathcal{O}_X) \overset{\mathrm{vol}_X}{\simeq} \mathrm{R\Gamma}(X,\Omega^3_X) \xrightarrow{\mathrm{tr}_{Gr}} k[-3].$$

(ii) Then, one considers the stack $\mathrm{Perf}^{\mathcal{O}_X[+1]}(X)$ consisting of perfect complexes of virtual rank 1 and with trivialized determinant. The (-1)-shifted symplectic form on $\mathrm{Perf}(X)$ *restricts* to a closed 2-form on $\mathrm{Perf}^{\mathcal{O}_X[+1]}(X)$ which is still non-degenerate. This has the effect of simply restricting to the *traceless* part of RHom in the previous displayed equation

(1) $$\mathrm{RHom}_X(\mathscr{E},\mathscr{E})_0^{\otimes 2} \xrightarrow{\mathrm{tr}} \mathrm{R\Gamma}(X,\mathcal{O}_X) \overset{\mathrm{vol}_X}{\simeq} \mathrm{R\Gamma}(X,\Omega^3_X) \xrightarrow{\mathrm{tr}_{Gr}} k[-3].$$

and noting that this resulting pairing is still non-degenerate. (Note that this last assertion relies on the fact that $\mathscr{E}$ had virtual rank 1, so that the traceless part is a complement to identity maps.)

(iii) Finally, stable pairs form an open substack of $\mathrm{Perf}^{\mathcal{O}_X[+1]}(X)$ – namely, we require that our perfect complex actually be a sheaf.

Note that in this case the only new argument, beyond [PTVV], was in step (ii) and it concerned only the *non-degeneracy* of an already defined closed 2-form in the right degree. In particular, we didn't have to define anything new – merely verify a linear algebraic property.

In the case that X is not assumed proper, an obvious complication arises: Since X is non-proper, we cannot expect to integrate a closed 2-form on $X \times \mathrm{Perf}(X)$ down to $\mathrm{Perf}(X)$. Moreover since $\mathrm{RHom}_X(\mathscr{E}, \mathscr{E})$ will generally be infinite-dimensional (in our desired application, it contains $\mathrm{R}\Gamma(X, \mathcal{O}_X)$ as a summand!) there's clearly no hope for Step (i) above.

Let us first explain how, in our desired application, this linear-algebra level problem is resolved in the case of X non-proper. It will turn out that the sheaf of traceless endomorphisms $\mathcal{RH}om_X(\mathscr{E}, \mathscr{E})_0$ has *proper support*, so that Grothendieck duality for sheaves with proper support provides a non-degenerate pairing generalizing that of Equation 1. It remains to define a closed two-form having this pairing as its underlying two-form. The construction is fairly simple:

(i) In Section 4, we will observe that in our desired example there is a large open locus $\mathfrak{U} \subset \mathrm{Perf}^{\mathscr{L}}(X)$ such that the natural closed 2-form on $X \times \mathfrak{U}$ has *proper support* over $\mathfrak{U}$;

(ii) In Section 3, we will discuss how to integrate down closed forms with compact support.

2. Reminder on PTVV: Integration maps

We remind the reader of some constructions in [PTVV], although we will use different notation.

Definition 2.0.1. Suppose that $\mathscr{X} = \mathrm{Spec}\, A$ is an affine derived pre-stack over k. Define the filtered chain complex

$$F^k C^\bullet_{dR}(\mathscr{X})$$

as the formal completion of k along the morphism of commutative dg algebras $k \to A$ together with its resulting "I-adic" filtration.[1] Recall that there is an identification,

$$\mathrm{gr}^k_F C^\bullet_{\mathrm{dR}}(\mathscr{X}) \simeq \mathrm{Sym}^k(\mathbb{L}_{\mathscr{X}}[-1]).$$

Note that any morphism $f\colon \mathscr{X} = \mathrm{Spec}\, A \to \mathscr{X}' = \mathrm{Spec}\, A'$ induces a diagram of k-algebras $A' \to A$ and thus a map on completions. So, there are restriction morphisms

$$f^*\colon F^k C^\bullet_{dR}(\mathscr{X}') \to F^k C^\bullet_{\mathrm{dR}}(\mathscr{X}')$$

for each k.

Definition 2.0.2. If $\mathscr{X} = \mathrm{Spec}\, A$ is an arbitrary derived pre-stack over k, define

$$F^k C^\bullet_{dR}(\mathscr{X}) = \lim_{U = \mathrm{Spec}\, A \to \mathscr{X}} F^k C^\bullet_{dR}(U).$$

With this, [PTVV, Def. 2.3] observes that:

Proposition 2.0.3. *Suppose that $\mathscr{F}$ is any derived pre-stack, and that $\mathscr{X}$ is an $\mathcal{O}$-compact derived stack in the sense of op.cit.. Then, there is a natural (in $\mathscr{F}$) morphism*

$$F^k C^\bullet_{dR}(\mathscr{X} \times \mathscr{F}) \longrightarrow \mathrm{R}\Gamma(\mathscr{X}, \mathcal{O}_X) \otimes F^k C^\bullet_{dR}(\mathscr{F})$$

Composing with a choice of morphism $\eta\colon \mathrm{R}\Gamma(X, \mathcal{O}_X) \to k[-d]$ gives rise to an "integration" map

$$\int_{[X]_\eta} : F^k C^\bullet_{dR}(\mathscr{X} \times \mathscr{F}) \longrightarrow F^k C^\bullet_{dR}(\mathscr{F})[-d].$$

Idea. Using that $\mathscr{X}$ is $\mathcal{O}$-compact, one can reduce to the case of both $\mathscr{X}$ and $\mathscr{F}$ affine. In this case, one composes an inverse to the Kunneth isomorphism

$$\boxtimes\colon C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, A) \otimes C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, B) \xrightarrow{\sim} C^\bullet_{\mathrm{dR}}(\mathrm{Spec}(A \otimes B))$$

with the projection

$$\eta\colon C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, A) \to C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, A)/F^1 C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, A) \simeq A$$

to obtain a morphism

$$\eta \circ \boxtimes^{-1}\colon C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, A) \otimes C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, B) \longrightarrow A \otimes C^\bullet_{\mathrm{dR}}(\mathrm{Spec}\, B). \qquad \square$$

[1] This is just a roundabout description of the "derived de Rham complex." Throughout this note we will implicitly use various formal properties of the derived de Rham complex that are well-known for ordinary de Rham cohomology a la Hartshorne.↑

2.0.4. The prototypical application is to the case where $\mathscr{X}$ is a d-CY proper variety with given volume form $\mathrm{vol}_{\mathscr{X}}$. Then, η is the composite of the volume form with the Grothendieck trace map map for $\mathscr{X}$

$$\mathrm{R}\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \xrightarrow{\mathrm{vol}_{\mathscr{X}}} \mathrm{R}\Gamma(\mathscr{X}, \Omega^d_{\mathscr{X}}) \xrightarrow{\mathrm{tr}} k[-d].$$

3. Compactly supported integration

Definition 3.0.5. Suppose that $\mathscr{X}$ is a derived pre-stack and $K \subset \mathscr{X}$ a closed subset. Define the (filtered) chain complex of relative de Rham cochains to be

$$F^k C^\bullet_{dR}(\mathscr{X}, \mathscr{X} \setminus K) \stackrel{\text{def}}{=} \mathrm{fib}\left\{ i^* \colon F^k C^\bullet(\mathscr{X}) \to F^k C^\bullet(\mathscr{X} \setminus K) \right\}.$$

Note that an inclusion $K \subset K'$ gives rise to an inclusion the other way of open complements $\mathscr{X} \setminus K' \subset \mathscr{X} \setminus K$, and thus to a *covariant* map

$$F^k C^\bullet_{dR}(\mathscr{X}, \mathscr{X} \setminus K) \longrightarrow F^k C^\bullet(\mathscr{X}, \mathscr{X} \setminus K')$$

on relative de Rham cochains. The compactly supported de Rham cochains are defined as the directed colimit

$$F^k C^\bullet_{c,dR}(\mathscr{X}) = \varinjlim_{\substack{K \subset \mathscr{X} \\ proper}} F^k C^\bullet_{dR}(\mathscr{X}, \mathscr{X} \setminus K).$$

over all closed subsets $K \subset \mathscr{X}$ which are proper over the base. If $\mathscr{X}$ is an S-prestack, then we can define a relative variant

$$F^k C^\bullet_{c/S,dR}(\mathscr{X}) = \varinjlim_{\substack{K \subset \mathscr{X} \\ \text{proper over } S}} F^k C^\bullet_{dR}(\mathscr{X}, \mathscr{X} \setminus K).$$

Finally, let us record the integration result that we will need:

Theorem 3.0.6. *Suppose that X is a smooth d-dimensional scheme, and that $\mathscr{F}$ is a derived pre-stack almost of finite presentation over k. A choice of volume form $\mathrm{vol}_X \colon \mathcal{O}_X \to \Omega^d_X$ gives rise to an integration map of filtered complexes*

$$\int_{[X]} \mathrm{vol}_X \wedge - \colon F^\bullet C^\bullet_{c/\mathscr{F},dR}(X \times \mathscr{F}) \longrightarrow F^\bullet C^\bullet_{dR}(\mathscr{F})[-d]$$

such that the induced map on associated graded pieces is the Grothendieck-Serre trace map.

Remark 3.0.7. In the argumented recalled in the previous subsection, [PTVV] was able to only ever use the absolute trace map for X – for which [H1] is a suitable reference exist. In the present situation, we need to use *relative* trace maps along the non-proper morphism $p_2 \colon X \times \mathscr{F} \to \mathscr{F}$ for sheaves having support proper over $\mathscr{F}$ – already here, it seems that a reference does not yet exist in the literature.

We do not wish to dwell on this point here, so we will instead collect the facts that we need in the Appendix.

Proof. Notice that both sides transform colimits in $\mathscr{F}$ to inverse limits of complexes, so that we can reduce to the case where $\mathscr{F} = Y = \operatorname{Spec} A$ is assumed affine.

Let $\widehat{\Omega^\bullet_Y}$ (resp., $\widehat{\Omega^\bullet_X}$, $\widehat{\Omega^\bullet_{X\times Y}}$) denote the sheaf of filtered complexes on Y (resp., X, $X \times Y$) that is the derived de Rham cochains on Y (resp., X, $X \times Y$).

Then, there is a Kunneth morphism of sheaves of filtered complexes on $X \times Y$

$$\boxtimes \colon \widehat{\Omega^\bullet_X} \boxtimes \widehat{\Omega^\bullet_Y} \longrightarrow \widehat{\Omega^\bullet_{X\times Y}}$$

which induces an isomorphism on associated gradeds. Furthermove, the assertion in [PTVV] implies that it is in fact a quasi-isomorphism of sheaves of filtered complexes on $X \times Y$ – in fact, for our purposes it is enough to note only that the right-hand side is the completion of the left.

Next, recall that there is an evident projection morphism

$$p \colon \widehat{\Omega^\bullet_X} \longrightarrow \widehat{\Omega^\bullet_X}/F^1 \simeq \mathcal{O}_X$$

Thus we can consider the following composite morphism of sheaves of filtered complexes on $X \times Y$:

$$\mathrm{vol}_X \wedge - \colon \widehat{\Omega^\bullet_X} \boxtimes \widehat{\Omega^\bullet_Y} \xrightarrow{p \otimes \mathrm{id}} \mathcal{O}_X \boxtimes \widehat{\Omega^\bullet_Y} \xrightarrow{\mathrm{vol}_X \otimes \mathrm{id}} \Omega^d_X \boxtimes \widehat{\Omega^\bullet_Y}[+d]$$

Notice that this is a morphism of sheaves of filtered complexes, where we must shift the filtration by d. Now Cor. A.0.12 guarantees the existence of the resulting map, and the discussion in the Appendix recalls what we mean by the Grothendieck-Serre trace map.

For the reader who does not want to read the Appendix, but wants an impression of what's going on, we will provide refernces to the classic literature in the case that X and Y are smooth classical schemes. In this case, the relevant construction is accomplished in [H1, II.2] using the Cousin-type "canonical resolution" of de Rham complexes. In this case, by the Grothendieck-Serre trace map for the (shifted) vector bundle

$$\Omega_X^d[+d] \boxtimes \mathrm{gr}_\ell \widehat{\Omega_Y^\bullet} \simeq \Omega_X^d \boxtimes \Omega_Y^\ell[d-\ell]$$

we mean a shift of the one constructed in [H1] using residual complexes. Then, [H2, Prop. 2.2] is devoted to verifying its compatibility with the de Rham differential and [H2, Prop. 2.3] observes the well-definedness of the trace map when restricting to proper supports. ☐

4. Variations on Perf X

In [PTVV] it is shown that if X is a proper d-CY variety then Perf X carries a $(2-d)$-symplectic structure. In enumerative applications, one often wants the following variant:

Theorem 4.0.8. *Suppose that X is a variety and that $\mathscr{L} = \mathcal{O}_X[+d] \in \mathrm{Pic}^{gr} X$ is the trivial line in grading $d \neq 0$. Let*

$$\mathrm{Perf}^{\mathscr{L}}(X) = \mathrm{Perf}(X) \times_{\mathrm{Pic}^{gr}(X)} \{\mathscr{L}\}$$

be the stack of perfect complexes on X with determinant fixed to be $\mathscr{L}$ (i.e., virtual rank $+d$ and trivial determinant).

Let $\mathfrak{U} \subset \mathrm{Perf}^L(X)$ be an open sub-stack satisfying the following properness condition:

(P) For any ring R and R-point $\eta\colon \operatorname{Spec} R \to \mathfrak{U}$, let $\mathscr{F} \in \mathrm{Perf}(X_R)$ be the perfect complex classified by η. Then, we require that the cone of the trace map of sheaves on $X_R := X \times \operatorname{Spec} R$

$$\mathcal{RH}om_X(\mathscr{F}, \mathscr{F}) \xrightarrow{\mathrm{tr}} \mathcal{O}_X$$

have support proper over $\operatorname{Spec} R$.

Then, $\mathfrak{U} \subset \mathrm{Perf}^L(X)$ carries a $(2-d)$-symplectic structure. Furthermore, this is natural for open inclusions of substacks satisfying the above condition.

Proof. We will imitate the proof in [PTVV] in four steps.

Step 1: *Pull back the universal form from* Perf *to $X \times \mathfrak{U}$:*
Begin by considering the commutative diagram

$$\begin{array}{ccccccc} X_{\mathfrak{U}} = X \times \mathfrak{U} & \overset{j}{\hookrightarrow} & X \times \mathrm{Perf}^{\mathscr{L}}(X) & \xrightarrow{i} & X \times \mathrm{Perf}(X) & \xrightarrow{ev} & \mathrm{Perf} \\ & & \downarrow & & \downarrow & & \\ & & \{\mathscr{L}\} & \longrightarrow & \mathrm{Pic}^{gr}(X) & & \end{array}$$

of derived pre-stacks. Let $\mathscr{E} \in \mathrm{Perf}(\mathrm{Perf})$ be the universal perfect complex and

$$\omega_{\mathrm{Perf}} = \mathrm{ch}(\mathscr{E})_2 \in H_0\left(F_2 C_{dR}^\bullet(\mathrm{Perf})[+2]\right)$$

be the 2-symplectic form constructed in [PTVV].

Recall that there is a well-defined pullback on derived de Rham complexes and that this respects the filtration. Thus, we obtain a class

$$\omega_{X\times\mathfrak{U}} = j^* i^* \mathrm{ev}^*(\omega_{\mathrm{Perf}}) \in H_0\left(F_2 C_{dR}^\bullet(X \times \mathfrak{U})[+2]\right)$$

and by functoriality of the Chern character, we have

$$\omega_{X\times\mathfrak{U}} = \mathrm{ch}(j^* i^* \mathrm{ev}^* \mathscr{E})_2 \in H_0\left(F_2 C_{\mathrm{dR}}^\bullet(X \times \mathfrak{U})\right)[+2].$$

Step 2: *Lift the form to cochains compactly supported over $\mathfrak{U}$:*
Let

$$\mathscr{F} = j^* i^* \mathrm{ev}^* \mathscr{E} \in \mathrm{Perf}(X \times \mathfrak{U})$$

and let $\mathscr{K} \subset X \times \mathfrak{U}$ be the support of the cone of

$$\mathcal{RH}om_{X\times\mathfrak{U}}(\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{tr}} \mathcal{O}_{X\times\mathfrak{U}}$$

The assumption (P) on $\mathfrak{U}$ is precisely the requirement that K be proper over $\mathfrak{U}$. We claim that we can lift $\omega_{X\times\mathfrak{U}}$ to an element of the relative group

$$F^2C^\bullet_{\mathrm{dR}}(X\times\mathfrak{U}, X\times\mathfrak{U}\setminus K)[+2]$$

Equivalently, setting $V = X\times\mathfrak{U}\setminus K$ we claim that we can canonically trivialize the restriction

$$\omega_{X\times\mathfrak{U}}|_V = \mathrm{ch}(\mathscr{F}|_V)_2 \in H_0\left(F_2C^\bullet_{\mathrm{dR}}(V)\right)$$

But indeed, on V the trace provides an isomorphism

$$\mathrm{tr}\colon\ \mathrm{RHom}_V(\mathscr{F}|_V, \mathscr{F}|_V) \xrightarrow{\sim} \mathcal{O}_V$$

so that $\mathscr{F}|_V$ is invertible under $\otimes_V$ and thus is a graded line. Consequently, there are natural isomorphisms

$$\mathscr{F}|_V \simeq (\det\mathscr{F})|_V \simeq \det(\mathscr{F}|_V) \simeq \mathcal{O}_V[+d]$$

giving rise to a trivialization

$$\mathrm{ch}(\mathscr{F}|_V)_2 = \mathrm{ch}(\mathcal{O}_V[+d])_2 = 0.$$

Step 3: *Integrate the form down to $\mathfrak{U}$:*
We saw in Step 2 that $\omega_{X\times\mathfrak{U}}$ lifts to a relative group, and thus to the group of cochains compactly supported over $\mathfrak{U}$:

$$\omega_{X\times\mathfrak{U}} \in H_0\left(F^2C^\bullet_{c/\mathfrak{U},\mathrm{dR}}(X\times\mathfrak{U})[+2]\right)$$

Applying Theorem 3.0.6, we can integrate this to obtain

$$\omega_{\mathfrak{U}} \stackrel{\mathrm{def}}{=} \int_{[X]} \omega_{X\times\mathfrak{U}} \in H_0\left(F^2C^\bullet_{dR}(\mathfrak{U})[+2-d]\right)$$

Step 3: *Check that the integrated form is non-degenerate.*
It remains to check that the image of $\omega_{\mathfrak{U}}$ in $H_0(\mathrm{gr}^2_F(\cdots))$ induces a non-degenerate pairing on $T_{\mathfrak{U}}$. Unravelling the definitions, as in [PTVV], we see the following: Fix an R-point $\mathrm{Spec}\, R \to \mathfrak{U}$ classifying a perfect complex $\mathscr{E} \in \mathrm{Perf}(X_R)$ equipped with an identification $\det\mathscr{E} \simeq (\mathscr{L})_R$; the tangent space at this point is

$$\mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E})_0 = \mathrm{fib}\left\{\mathrm{tr}\colon\ \mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E}) \longrightarrow \mathrm{R}\Gamma(X_R,\mathcal{O}_{X_R})\right\}.$$

Let K be the support of the similarly defined $\mathcal{RH}om_{X_R}(\mathscr{E},\mathscr{E})_0$. By assumption (P), K is proper over R.

Recall that there is a composite map

(2) $$\mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E})_0^{\otimes 2} \longrightarrow \mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E})^{\otimes 2} \xrightarrow{m} \mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E}) \xrightarrow{\mathrm{tr}} \mathrm{R}\Gamma(X_R,\mathcal{O}_{X_R})$$

Since $\mathcal{RH}om_{X_R}(\mathscr{E},\mathscr{E})_0^{\otimes 2}$ has support in K, this composite map factors canonically through a map

$$\mathrm{RHom}_{X_R}(\mathscr{E},\mathscr{E})_0^{\otimes 2} \longrightarrow \mathrm{R}\Gamma_K(X_R,\mathcal{O}_{X_R})$$

Observe that the pairing induced by $\omega_{\mathfrak{U}}$ is precisely the composite of this map, with the composite of the volume form on X and the Grothendieck trace map

$$\eta_K = \mathrm{tr}_p \circ \mathrm{vol}_X\colon\ \mathrm{R}\Gamma_K(X_R,\mathcal{O}_{X_R}) \xrightarrow{\mathrm{vol}_X} \mathrm{R}\Gamma_K(X_R,\Omega^d_X\boxtimes\mathcal{O}_{\mathrm{Spec}\,R}) \xrightarrow{\mathrm{tr}_p} R[-d]$$

where tr_p is as in Theorem A.0.10. To complete the proof, we now make two observations:

(i) Note that if $\mathscr{E} \in \mathrm{Perf}\, X$ has virtual dimension d, then $\mathrm{tr}(\mathrm{id}_{\mathscr{E}}) = d$. Since $d \neq 0$, we see that the trace map is split by $\frac{1}{d}\,\mathrm{id}_{\mathscr{E}}$ so that we have a direct sum decomposition of sheaves

$$\mathcal{RH}om_{X_R}(\mathscr{E},\mathscr{E}) \simeq \mathcal{RH}om_{X_R}(\mathscr{E},\mathscr{E})_0 \oplus \mathcal{O}_{X_R}$$

which, by construction, is orthogonal with respect to the trace pairing. It follows that the trace pairing induces a perfect pairing of sheaves

$$\mathcal{RH}om_{X_R}(\mathscr{E},\mathscr{E})_0^{\otimes 2} \longrightarrow \mathcal{O}_{X_R}.$$

Note that taking global sections precisely recovers the composite map of Equation 2.

(ii) It is thus enough to show the following: Suppose $\mathrm{ev}\colon \mathscr{F} \otimes \mathscr{F}^* \to \mathcal{O}_{X_{\mathscr{R}}}$ is the canonical perfect pairing for $\mathscr{F} \in \mathrm{Perf}(X_{\mathscr{R}})$ having support in K. The map on global sections thus factors through $\mathrm{R}\Gamma_K(X_R, \mathcal{O}_{X_R})$ and we must show that the resulting pairing

$$\mathrm{R}\Gamma(X_R, \mathscr{F}) \otimes \mathrm{R}\Gamma(X_R, \mathscr{F}^*) \longrightarrow \mathrm{R}\Gamma_K(X_R, \mathcal{O}_{X_R}) \xrightarrow{\eta_K} R[-d]$$

is a perfect pairing. This is the form of Grothendieck-Serre duality appearing in Cor. A.0.11. □

Finally, we have our motivating example:

Corollary 4.0.9. *Suppose that X is a (not necessarily compact) 3-CY variety and $\mathscr{L} = \mathcal{O}_X[+1]$, so that $\mathrm{Perf}^{\mathscr{L}}(X)$ consists of perfect complexes of virtual dimension $+1$ equipped with a trivialization of their determinant. Let $\mathfrak{U} \subset \mathrm{Perf}^L(X)$ be the locus classifying ideal sheaves of proper subvarieties. Then, $\mathfrak{U}$ satisfies the conditions of the previous Theorem.*

Proof. Suppose that $\eta\colon \operatorname{Spec} R \to \mathfrak{U}$ is a point, corresponding to $\mathscr{E} \in \mathrm{Perf}(X_R)$ together with an identification $\det(\mathscr{E}) \simeq \mathcal{O}_X[+1]$ and a guarantee that $\mathscr{E}$ is an ideal sheaf. In this case, the natural map

$$\mathscr{E} \longrightarrow (\mathscr{E}^*)^* \simeq (\det \mathscr{E})[-1] \simeq \mathcal{O}_X$$

exhibits it as an ideal sheaf, with the cone having proper support by assumption. This shows that the trace map

$$\mathrm{tr}\colon\ \mathrm{RHom}(\mathscr{E}, \mathscr{E}) \longrightarrow \mathcal{O}_X$$

is an isomorphism away from this proper support as well. □

Appendix A. Some facts on Grothendieck-Serre duality

We will need the following case of Grothendieck-Serre duality in the derived context.

Theorem A.0.10. *Suppose that k is a field; that X is a quasi-compact, separated, and smooth k-scheme of dimension d; and, that Y is an arbitary k-scheme. Let $p\colon X \times_k Y \to Y$ denote the projection, and suppose that $K \subset X \times Y$ is a closed subscheme that is proper over Y.*

Then, there exists a relative trace morphism

$$\mathrm{tr}_p\colon p_* \circ \underline{\mathrm{R}\Gamma_K}(\omega_p) \longrightarrow \mathcal{O}_Y \quad \text{where} \quad \omega_p \stackrel{def}{=} \Omega^d_X[+d] \boxtimes \mathcal{O}_Y$$

which is

(i) Compatible with arbitrary base-change in Y; and,

(ii) Suppose that $\mathscr{P} \in \mathrm{Perf}(X \times Y)$ is set-theoretically supported on K, and let $\mathscr{P}^\vee = \mathcal{RH}om_{X\times Y}(\mathscr{P}, \mathcal{O}_{X\times Y})$ be its dual. The evaluation morphism

$$\mathscr{P} \otimes_{X\times Y} (\mathscr{P}^\vee \otimes \omega_p) \longrightarrow \omega_p$$

factors uniquely through $\underline{\mathrm{R}\Gamma_K}(\omega_p)$ and induces a map

$$p_*(\mathscr{P}) \otimes_{\mathcal{O}_S} p_*(\mathscr{P}^* \otimes \omega_p) \longrightarrow p_* \underline{\mathrm{R}\Gamma_K}(\omega_p) \xrightarrow{\mathrm{tr}_p} \mathcal{O}_Y.$$

We require this to be a perfect pairing of quasi-coherent $\mathcal{O}_Y$-modules, whenever Y is the spectrum of a field.

(iii) For any $\mathscr{F} \in \mathrm{QC}(Y)$, there is an induced map

$$\mathrm{tr}_{p,\mathscr{F}}\colon p_* \circ \underline{\mathrm{R}\Gamma_K}(\Omega^d_X[+d] \boxtimes \mathscr{F}) \longrightarrow \mathscr{F}$$

by using the projection formula and tensoring tr_p with the identity on $\mathscr{F}$. We require that this be compatible with finite pushforward in Y.

Let us deduce two Corollaries of the Theorem:

Corollary A.0.11. *The pairing in Theorem A.0.10(ii) is a perfect pairing for arbitrary Y.*

Proof. The hypotheses on X and Theorem A.0.10(i) guarantee that the formation of $p_*(\mathscr{P})$, $p_*(\mathscr{P}^\vee \otimes \omega_p)$, and the pairing are compatible with arbitary base-change on Y. Next, note that each of $p_*(\mathscr{P})$ and $p_*(\mathscr{P}^* \otimes \omega_p)$ are almost perfect and, since p is flat, of finite Tor amplitude – thus they are perfect. Consequently, the sheaf Hom

$$\mathcal{RH}om_{\mathcal{O}_Y}(p_*(\mathscr{P}), \mathcal{O}_Y)$$

is quasi-coherent and in fact perfect (similarly for $p_*(\mathscr{P}^\vee \otimes \omega_p)$). Thus, Nakayama's Lemma applies – and we are reduced to checking that the pairing is perfect after base-change to each closed point of Y. This is guaranteed by Theorem A.0.10(ii). □

Corollary A.0.12. *Let $\widehat{\Omega^\bullet_Y}$ denote the sheaf of filtered complexes on Y of derived de Rham complexes. Then, there is a trace morphism*

$$p_* \circ \underline{\mathrm{R}\Gamma_K}(\Omega^d_X[+d] \boxtimes \widehat{\Omega^\bullet_Y}) \longrightarrow \widehat{\Omega^\bullet_Y}$$

such that the induced map on the ℓ-th associated graded piece homotopic to $\mathrm{tr}_{p,\mathrm{Sym}^\ell \mathbb{L}_Y[-1]}$ (here we implicitly use the identification of the ℓ-th graded piece with this quasi-coherent sheaf.)

Proof. In the case where Y is a smooth scheme, this is checked for the classical trace map in [H2, Prop. 2.2]. Since we are trying to avoid committing to a particular model for the trace morphisms, our argument will be different. We will take advantage of re-formulation of the de Rham complex via the mixed complexes in [TV], as follows:

Denote the unipotent loop scheme by $L^uY = \mathrm{Map}(B\mathbb{G}_a, Y)$, so that L^u is acted on by the derived group scheme $G = \mathbb{G}_m \ltimes B\mathbb{G}_a$ – this is a somewhat geometric re-formulation of the mixed complex structure on $\mathcal{O}_{L^uY}$. Notice that there is a natural "inclusion of constant loops" map $Y \to L^uY$, and that it realizes L^uY as a nilpotent thickening of Y. In particular, we may regard K as a subset of $X \times LY$ that is proper over LY.

From [TV], we conclude that it is enough to give a map of G-equivaraint complexes (i.e., mixed complexes)

$$\mathrm{R}\Gamma\left(L^uY, p_* \circ \underline{\mathrm{R}\Gamma_K}(\Omega^d_X[+d] \boxtimes \mathcal{O}_{L^uY})\right) \longrightarrow \mathrm{R}\Gamma\left(L^uY, \mathcal{O}_{L^uY}\right)$$

that on the ℓ-th graded piece is what we expect it to be. We make three remarks:

(i) Theorem A.0.10 implies the existence of such a trace morphisms;
(ii) The compatibility with base-change implies that the trace map lifts to a G-equivariant one: for every $\mathrm{Spec}\, R \to BG$, we can consider $LY_R = (LY/G) \times_{BG} \mathrm{Spec}\, R$ and the projection $X \times LY_R \to LY_R$ and base-change guarantees that this will be a quasi-coherent complex lifting the expected one;
(iii) There is a $\mathbb{G}_m$-equivariant (but not G-equivariant) projection morphism $L^uY \to Y$. This morphism is finite, and applying Theorem A.0.10(iii) yields the identification of the induced morphism on associated graded pieces. □

Finally, we make some remarks on the proof of Theorem A.0.10:

Remarks / Proof Sketch for Theorem A.0.10.

(i) If Y is classical, then this is discussed in [H1] and [C]. The proofs use residual complexes, which seem fundamentally less convenient with the passage to derived schemes.
(ii) If a "feature complete" theory of Grothendieck-Serre duality in the derived setting (but without supports) were available, it would be possible to deduce this version from it by a limiting argument (replacing $X \times Y$ by the colimit of all $Z \to X \times Y$ which are proper over Y and set-theoretically factor through K) – we expect a reference for such a theory to appear in upcoming work of Gaitsgory-Rozenblyum.
(iii) If $k = \mathbb{C}$ and if Y is also assumed of finite-type over k, then it is likely possible to define this map via a Dolbeault resolution of $\Omega^d_X[+d]$ and an analytic integration map. This would, however, depend on a good theory of derived analytic spaces.
(iv) Here is a rather different Proof Sketch, in the spirit of Lipman's development of the residue [L] and in the spirit of many parts of [PTVV]. It also comes close to proving the second Corollary along the way, so perhaps it is the "right" proof for our purposes. Nevertheless, in light of (i) and (ii) above we leave it as only a sketch:

The assertion is local, so that we may suppose that Y is affine. In this case, each of X, Y, and $X \times Y$ are "perfect stacks" in the sense of [BZFN]. From this one can deduce that the pre-sheaf of complexes on $X \times Y$ determined by

$$U \mapsto \mathbf{HH}_\bullet(\operatorname{Perf} U)$$

is in fact a sheaf. We will denote this sheaf by $\underline{\mathbf{HH}_\bullet}(X \times Y)$, and analogously for $\underline{\mathbf{HH}_\bullet}(Y)$ on Y.

We first construct a morphism of sheaves of complexes on Y

$$\operatorname{tr}' \colon p_* \underline{\mathrm{R}\Gamma_K}(\underline{\mathbf{HH}_\bullet}(X \times Y)) \longrightarrow \underline{\mathbf{HH}_\bullet}(Y)$$

and then define tr_p to be the composition of this with the (Hochschild-Kostant-Rosenberg related) inclusion-type map

$$\Omega^d_X[+d] \boxtimes \mathcal{O}_{LY} \longrightarrow \underline{\mathbf{HH}_\bullet}(X) \boxtimes \underline{\mathbf{HH}_\bullet}(Y) \xrightarrow{\sim} \underline{\mathbf{HH}_\bullet}(X \times Y)$$

and the projection-type map

$$\underline{\mathbf{HH}_\bullet}(Y) \simeq \mathcal{O}_{LY} \longrightarrow \mathcal{O}_Y.$$

(We remark that tr' is a close relative to the map appearing in the proof of the previous Corollary.)

To construct the morphism tr' we use the localization sequence

$$\operatorname{Perf}_K(X \times Y) \to \operatorname{Perf}(X \times Y) \to \operatorname{Perf}(X \times Y \setminus K)$$

which implies that there is an identification

$$\mathbf{HH}_\bullet(\operatorname{Perf}_K(X \times Y)) \xrightarrow{\sim} \mathrm{R}\Gamma_K(\underline{\mathbf{HH}_\bullet}(X \times Y))$$

and a corresponding sheafy version that we omit. Thus, the existence of the morphism tr' follows from the functoriality of Hochschild homology and the proper pushforward theorem – the latter guarantees that p_* takes $\operatorname{Perf}_K(X \times Y)$ to $\operatorname{Perf}(Y)$ since p has finite Tor dimension.

To prove the compatibility of tr_p with base change, it is enough by construction to prove the analogous statement for tr'. To deduce this from from standard properties of Hoschchild homology, it is enough to show that the natural functor

$$\operatorname{Perf}_K(X \times Y) \otimes_{\operatorname{Perf} Y} \operatorname{Perf} Y' \longrightarrow \operatorname{Perf}_{K'}(X \times Y')$$

is an equivalence. This follows from the localization sequences and the tensor product theorem for Perf on perfect stacks.

To deduce the non-degeneracy in Theorem A.0.10(ii) we seem to need something non-formal to happen. By compatibility with base change, we may as well assume Y is the spectrum of a field. Now everything is classical, so that we can go backwards – it is enough to describe tr' in terms of the HKR isomorphism, the Atiyah class of X, and the ordinary Grothendieck-Serre trace maps, and to see that the ordinary trace map is the indicated component. Without the proper support condition, such a computation occurs in [M1].

Finally, to obtain the compatibility Theorem A.0.10(iii) we re-interpret $\operatorname{tr}_{p,\mathscr{F}}$ as arising from a functorial construction as well: Regard $- \otimes_{\mathcal{O}_Y} \mathscr{F}$ as an endo-functor of $\mathrm{QC}(Y)$, and $- \otimes_{\mathcal{O}_{X \times Y}} p^*\mathscr{F}$ as an endo-functor of $\mathrm{QC}(X \times Y)$ preserving $\mathrm{QC}_K(X \times Y)$. The projection formula makes precise the compatibility between them, so that there is an induced map

$$\operatorname{tr}'_{\mathscr{F}} \colon \mathbf{HH}_\bullet(\mathrm{QC}_K(X \times Y), p^*\mathscr{F}) \longrightarrow \mathbf{HH}_\bullet(\mathrm{QC}(Y), \mathscr{F})$$

and one can describe $\operatorname{tr}_{p,\mathscr{F}}$ in terms of $\operatorname{tr}'_{\mathscr{F}}$. Then, it is enough to prove a similar compatibility for $\operatorname{tr}'_{\mathscr{F}}$ by formal non-sense. □

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