IND-COHERENT COMPLEXES ON LOOP SPACES AND CONNECTIONS

ANATOLY PREYGEL
JANUARY 28, 2017

Contents

1. Introduction

The goal of this note is to describe a functorial relationship between D-modules on X and quasi-coherent sheaves on the derived loop space LX, for X an algebraic space of finite type over a char. 0 field k. Note that X is *not necessarily assumed smooth*.

This is a generalization of work of Toën/Vezzosi and Nadler/Ben-Zvi to the case where X is not assumed smooth, with the added motivation that it is better compatible with the usual *functoriality* on D-modules. Indeed, the usual functors on D-modules satisfy the compatibilities (e.g., adjunctions) that one would expect of functors called $f^!$ and f_* – not f^* and f_* – so that it is desirable to compare them to sheaves on LX with similar functoriality. This will require us to replace quasi-coherent complexes with the Ind-coherent complexes of the title.

The present document will be halfway between a real paper and an announcement – we will try to state all the results, to survey the broad ideas that go into it, and to indicate the key pieces and steps in the proofs (often just by stating the Lemmas along the way). A detailed exposition, with full proofs, will appear in [P2].

1.1. **Decategorified results.** One motivation for this is to study the functoriality of the idea, due to Toen-Vezzosi and others, that R-linear categories roughly categorify D-modules on $\operatorname{Spec} R$. For instance, we may interpret functions on the loop space as the Hochschild cochains $\mathbf{HH}_\bullet(\operatorname{Perf} X)$.

Thus the work of Toen-Vezzosi and others can be viewed as categorifying, at least in the smooth case, the following beautiful result of Feigin and Tsygan:

Theorem 1.1.1 (Feigin-Tsygan)**.** *Suppose that X is an affine scheme of finite type over a char. 0 field k. Then, the k-linear periodic cyclic homology of X is naturally isomorphic to the $\mathbb{Z}/2$-graded infinitesimal cohomology of X:*

$$H_n \mathbf{HP}_\bullet^k(X) \simeq \oplus_{i\in\mathbb{Z}} H_{dR}^{n+2i}(X).$$

Furthermore, this identification can be made functorially with respect to pullback.

We will find it easier to work with the Grothendieck dual situation: We replace $\operatorname{Perf} X$ by $\operatorname{DCoh} X$. [P1, Appendix B] explains that $\mathbf{HH}_\bullet(\operatorname{DCoh} X)$ can be identified with the distributions on LX, etc. In this way, our categorical results will imply the following *Grothendieck/Poincaré dual* statement to Theorem 1.1.1 which we believe is of independent interest:

Theorem 1.1.2. *Suppose that X is a quasi-compact and separated algebraic space of finite type over a char. 0 field k. Let $\operatorname{DCoh} X$ denote the dg-category of quasi-coherent complexes on X with bounded coherent homology sheaves. Then, there is an identification of the k-linear periodic cyclic chains of $\operatorname{DCoh} X$ with the $\mathbb{Z}/2$-graded (de Rham) Borel-Moore chains on X:*

$$\mathbf{HP}_\bullet(\operatorname{DCoh} X) \simeq C_\bullet^{BM,dR}(X)_{\mathbb{Z}/2}$$

Furthermore, this identification can be made functorially with respect to quasi-smooth pullback and proper pushforward.

Notice that Theorem 1.1.2 serves as a test for the "functoriality" assertions – while de Rham cohomology is most easily seen as *endomorphisms*, the Borel-Moore chains are most easily seen as a pushforward (of the dualizing complex $f^! k$).

Also, let us make a notational remark: since all our proofs will go via derived algebraic geometry, for us a scheme / algebraic space / etc. is a *derived* scheme/ etc. unless explicitly called "classical" or "discrete."

1.2. **Main results – small categories.** In our formulation, the relationship will be defined completely naturally – without "formulas" – and depends on two contructions. The first is classical: We will think of D-modules as *crystals*, i.e., as sheaves on the de Rham space:

Definition 1.2.1. If X is a pre-stack, we let X_{dR} be the *de Rham* pre-stack of X:

$$X_{\mathrm{dR}}(R) := X((\pi_0 R)_{\mathrm{red}})$$

for all connective derived rings $R \in \mathbf{CAlg}$. This depends only on the underlying reduced, classical sub-pre-stack of X. Note also that there is a natural map $X \to X_{\mathrm{dR}}$.

1.2.2. If X is an algebraic space, then X and LX have the same underlying reduced, classical sub-pre-stack. In particular, $X_{\mathrm{dR}} \to (LX)_{\mathrm{dR}}$ is an equivalence. Thus, there is a natural SO(2)-equivariant diagram

$$LX \longrightarrow (LX)_{\mathrm{dR}} \simeq X_{\mathrm{dR}}$$

where the SO(2)-action on X_{dR} is the trivial one and the final equivalence is induced from the inclusion of constant looks $X \to LX$.

Second, we need the *Tate* construction for dg-categories[1] with SO(2)-action:

Definition 1.2.3. Suppose that $\mathcal{C} \in (\mathbf{dgcat}_k^{\mathrm{idm}})^{\mathrm{SO}(2)}$ is a small, stable, idempotent complete dg-category with SO(2)-action. Then, the invariant category

$$\mathcal{C}^{\mathrm{SO}(2)}$$

is again a small, stable, idempotent complete dg-category. But in fact it has extra structure – it is a module category over $(\operatorname{Perf} k)^{\mathrm{SO}(2)} = \operatorname{Perf} C^*(B\,\mathrm{SO}(2), k)$. We will once-and-for-all identify $C^*(B\,\mathrm{SO}(2), k) \simeq k[\![u]\!]$ as E_∞-algebras, where u has homological grading -2, and refer to $\mathcal{C}^{\mathrm{SO}(2)}$ as a $k[\![u]\!]$-linear category. Then,[2]

$$\mathcal{C}^{\mathrm{Tate}} := \mathcal{C}^{\mathrm{SO}(2)} \otimes_{k[\![u]\!]} k(\!(u)\!) \in \mathbf{dgcat}^{\mathrm{idm}}_{k(\!(u)\!)}.$$

The simplest form of our main result is

Theorem 1.2.4. *Suppose that X is an algebraic space almost of finite type over a characteristic-zero field k. Then, the natural* SO(2)*-equivariant map*

$$\pi \colon LX \to X_{\mathrm{dR}}$$

described above induces a natural equivalence of 2*-periodic (=$k(\!(u)\!)$-linear) dg-categories*

$$\pi_* \colon \operatorname{DCoh}(LX)^{\mathrm{Tate}} \xrightarrow{\sim} \operatorname{DCoh}(X_{\mathrm{dR}}) \otimes_k k(\!(u)\!)$$

where the right hand-side consists of $\mathbb{Z}/2$-graded coherent right crystals on X.

1.3. **Main results – large categories (with t-structures!)** In order to prove Theorem 1.1.2 from Theorem 1.2.4 one needs to do a bit of work because one needs more functoriality than DCoh can offer: Non-proper pushforward, and some form of $\pi^!$. Thus we want to pass to a bigger category: QC or (for reasons of functoriality) $\mathrm{QC}^! := \operatorname{Ind}\operatorname{DCoh}$.

We are then faced with the question of what the analog of the Tate construction should be. The naive guess would be that one should replace just imitate the definition above in the world of presentable ∞-categories. Unfortunately, one immediately sees that some extra care is required in defining the Tate construction even in case of $X = \mathrm{pt}$:

Example 1.3.1. Suppose that $X = \mathrm{pt}$ and thus $LX = X_{\mathrm{dR}} = \mathrm{pt}$ as well. Then,

$$\mathrm{QC}(X) = \operatorname{Ind}\operatorname{DCoh}(LX) = k\text{-mod}$$

One can produce $k[\![u]\!]$-linear equivalences of ∞-categories

$$(k\text{-mod})^{\mathrm{SO}(2)} \simeq H_*(\mathrm{SO}(2), k)\text{-mod} \simeq (k[\![u]\!]\text{-mod})^{u-nil}$$

where the super-script u-nil denotes the full subcategory of locally u-torsion modules.[3] In particular, one sees that naively imitating the definition for the Tate construction above – i.e., taking the tensor product in the sense of presentable ∞-categories – we have

$$(k\text{-mod})^{\mathrm{Tate}} \simeq (k[\![u]\!]\text{-mod})^{u-nil} \otimes_{k[\![u]\!]} k(\!(u)\!) = \{0\}.$$

At the other extreme, we could work in the world of compactly-generated categories and functors that preserve colimits *and* compact objects. This is too much, since non-proper pushforward and $\pi^!$ don't generally preserves the compact objects DCoh. Fortunately, one encounters a similar situation in building the functoriality for Ind DCoh and there is a fix using the t-structures!

1.3.2. Indeed, in order to recover $k(\!(u)\!)$-mod one must do something trickier. We were led to our preferred solution by [P1] where these same categories occured in a different way: Let $\Omega_0\mathbb{A}^1$ denote the derived fiber product $\mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt}$. One sees in op.cit. that $\mathrm{QC}(\mathbb{A}^1) \simeq (k[\![u]\!]\text{-mod})^{u-nil}$ while $\mathrm{QC}^!(\mathbb{A}^1) \simeq k[\![u]\!]\text{-mod}$. Certainly one way to bridge this gap is to restrict to some small subcategory – but another is to use the sort of t-structure trickery that is helpful in dealing with $\mathrm{QC}^!$. This trickery essentially amounts to murmuring the words "left t-exact up to a finite shift / left t-bounded" repeatedly, but turns out to be convenient.

---

[1] i.e., objects of the ∞-category $\mathbf{dgcat}_k^{\mathrm{idm}}$ of small, stable, idempotent complete Perf k-module categories↑

[2] When dealing with small stable idempotent-complete dg-categories, we take the notation $\mathcal{D} \otimes_R R'$ to be synonymous with $\mathcal{D} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R'$.↑

[3] i.e., A dg-module M is locally u-torsion if every map from a perfect complex becomes nullhomotopic after applying some power of u; equivalently, if every element of $\pi_* M$ is annihilated by some power of u.↑

Very roughly, the trick will be to restrict ourselves to categories that are compactly generated with t-structures that restrict to bounded t-structures on the compact objects ("regular"); and to functors that are colmit preserving and left t-exact (up to a finite shift). This allows enough functors that the pushforward and $\pi^!$ on $\mathrm{QC}^!$ are allowed, but few enough functors so that we still get the correct category of $\mathrm{SO}(2)$-invariants.

1.3.3. We'll discuss this trickery in more detail in Section 4: There we define the notions of (left) *complete*, *coherent*, and *regular* t-structures. For instance, $\mathrm{QC}(X)$ is complete for any reasonable stack X. If X is Noetherian, then both $\mathrm{QC}(X)$ and $\mathrm{QC}^!(X)$ are coherent and $\mathrm{QC}^!(X)$ is regular. Given a presentable ∞-category $\mathcal{C}$ with coherent t-structure there are two dual constructions – completion, and regularization: $\mathscr{R}(\mathcal{C}) \to \mathcal{C} \to \widehat{\mathcal{C}}$. Both $\widehat{\mathcal{C}}$ and $\mathscr{R}(\mathcal{C})$ can be recovered purely from the ∞-category $\mathcal{C}_{<0}$ but by rather different reconstruction procedures.

The formation of $\mathcal{C} \to \mathscr{R}(\mathcal{C})$ is easy to describe: One picks out $\mathrm{Coh}(\mathcal{C}) \subset \mathcal{C}$ to consist of the bounded objects $c \in \mathcal{C}$ such that their truncations $\tau_{\leq k} c \in \mathcal{C}_{\leq k}$ are compact for all k – in case $\mathcal{C} = R$-mod this picks out the bounded pseudo-coherent objects, which in the Noetherian case are precisely the bounded coherent complexes. Then, one sets $\mathscr{R}(\mathcal{C}) := \operatorname{Ind}\mathrm{Coh}(\mathcal{C})$. Section 4 explains that this is functorial for functors of $\mathcal{C}$ which preserve left t-bounded colimits and are (almost) left t-exact.

Definition 1.3.4. Suppose that $\mathcal{C}$ is a presentable ∞-category with coherent t-structure. Then, we define

$$\mathcal{C}^{t\mathrm{Tate}} := \mathscr{R}(\mathcal{C}^{\mathrm{SO}(2)}) \otimes_{k[\![u]\!]} k(\!(u)\!)$$

as $k(\!(u)\!)$-linear presentable ∞-category.

For instance, $\mathscr{R}(k\text{-mod}^{\mathrm{SO}(2)}) \simeq k[\![u]\!]$-mod so that $(k\text{-mod})^{t\mathrm{Tate}} \simeq k(\!(u)\!)$-mod. Notice, however, that the t-structures were only a tool to moderate the intermediate step (of taking invariants) and that final result cannot carry an interesting t-structure. It has the pleasant property of giving the same answers as does working with small categories, while having more evident functoriality.

The resulting form of our main result will then be:

Theorem 1.3.5. *Suppose that X is an algebraic space almost of finite type over a characteristic-zero field k. Then, the natural $\mathrm{SO}(2)$-equivariant map*

$$\pi \colon LX \to X_{\mathrm{dR}}$$

described above induces an adjoint pair of equivalence of 2-periodic (=$k(\!(u)\!)$-linear) dg-categories

$$\pi_* \colon \mathrm{QC}^!(LX)^{t\mathrm{Tate}} \xrightarrow{\sim} \mathrm{QC}^!(X_{\mathrm{dR}}) \otimes_k k(\!(u)\!) \colon \pi^!.$$

where the right-hand side consists of $\mathbb{Z}/2$-graded right crystals on X.

1.4. **Localization theorems and philosophy.** Perhaps more interesting than the Theorems themselves – though we think that they are interesting – is the philosophy behind them. The Theorems will be deduced from more general categorical "localization" theorems in the spirit of Atiyah-Bott Localization: Theorem 5.1.3 and Theorem 5.3.4. Ignoring all technical details, the proof of our Theorems (and, at this level of precision, of the Theorem of Feigin-Tsygan) would go as follows:

(1) Suppose that S is a topological space with an action of $G = \mathrm{SO}(2)$. Let $F \subset S$ denote the fixed point locus for this action. If S satisfies reasonable finiteness conditions, then the Atiyah-Bott Localization Theorem tells us that the map

$$C_\bullet(F) \otimes_k k(\!(u)\!) \longrightarrow C_\bullet(S)^{\mathrm{Tate}} := C_\bullet(S)^{\mathrm{SO}(2)} \otimes_{k[\![u]\!]} k(\!(u)\!)$$

is an equivalence of $k(\!(u)\!)$-module complexes.

(2) Suppose that $X = \operatorname{Spec} R$ is a scheme, assumed affine for simplicity. Then,

$$\mathrm{R}\Gamma(X^S, \mathcal{O}_{X^S}) = S \otimes R$$

is another (derived) commutative algebra, equipped with a map to R coming from the projection $S \to \mathrm{pt}$. So, we can consider the derived version of the I-adic filtration along the "kernel" of $S \otimes R \to R$. The upshot is a a filtration

$$F^\bullet(S \otimes R)$$

such that the associated graded can be identifies with

$$\mathrm{gr}_i F^\bullet(S \otimes R) \simeq \mathrm{Sym}^i_R \widetilde{C}_\bullet(S) \otimes_R \mathbb{L}_R$$

When this can be done equivariantly, (1) thus provides a "localization theorem" for each associated graded piece of the filtration. In characteristic zero, it can be shown that this filtration actually splits if S is a co-H-space. Based on this, it is reasonable to expect a localization theorem for the *completion* of $S \otimes R$ along this filtration. If $X = \operatorname{Spec} R$, then $\operatorname{Spec} S \otimes R$ is the derived mapping space $X^S := \mathrm{Map}(S, X)$ (see Notation Notation 5.1.1).

We switch now to this geometric language, letting X^S denote the derived mapping space and $\widehat{X^S}$ its completion along constant maps (these notions will be discussed more fully in Sections Section 2 and Section 5). Then, what we would like to ask for is a relationship like

$$\mathrm{R}\Gamma(\widehat{X^F}, \widehat{\mathcal{O}_{X^F}})^{\mathrm{Tate}} \xrightarrow{\sim} \mathrm{R}\Gamma(\widehat{X^S}, \widehat{\mathcal{O}_{X^S}})^{\mathrm{Tate}}$$

under some assumptions on F. (Our proof will, roughly, go along these lines. Although the Theorem of Feigin-Tsygan looks like an example of this sort of phenomenon, we do not prove it – the filtration does not split in this case, so more work is required.)

(3) We will see, as in [P1, GR1], that $\mathrm{QC}^!$ is very well adapted to studying formal completions by "descent." Thus, (2) will allow us to get categorical forms of the localization theorem – both in small and large category (with t-structure) forms.

(4) Finally, what's this got to do with loop spaces and connections? It turns out that we just have to consider the above in the case of

$$F = \emptyset \subset S^1 = S$$

with the usual action of $G = \mathrm{SO}(2)$. We will explain in Section 2 why X_{dR} is the right answer for the completion $\widehat{X^\emptyset} = \widehat{\mathrm{pt}}$ of the point along X.

1.5. **Plan of the paper.** We will spend the first few sections, Section 2, Section 3 and Section 4 and gathering some convenient general tools that are well suited to our problem:

- In Section 2 we will discuss formal completions, and the analog of the I-adic filtration, in the derived setting. One notable point is the observation that this filtration is *rationally split* in many cases, which we observe is a special case of general results on Goodwillie towers due to Bauer and McCarthy.
- In Section 3 we discuss some generalities on Ind-coherent compelxes, or $\mathrm{QC}^!(X)$ as we'll call the ∞-category of such, taken from [P1, G1].
- In Section 4 we discuss some general constructions on large categories with t-structures. These are implicit in many key constructions involving Ind-coherent complexes, and are used in our formulation for "large categories" above. We also advertise that these ideas have been gainfully used in [BZNP].

Then, in Section 5 we will state our general "Localization Theorems" and sketch their proofs using these tools. Finally, in Section 6 we provide a lower bound on how interesting the categorical results are – namely, we'll apply them to proving the decategorified Theorem 1.1.2.

1.6. **Notation and Clarifications.** We work throughout over a fixed field k of characteristic zero. All schemes / algebraic spaces / etc. are assumed to be derived, quasi-compact and quasi-separated, and of finite type over k unless explicitly stated otherwise. Similarly, the notation $\mathrm{QC}(X)$ or R-mod will refer to the ∞-category of quasi-coherent complexes on X or the ∞-category of R-modules in spectra or in k-module complexes. When we want to refer to the classical category, we will think of it as the heart of the t-structure, e.g., $\mathrm{QC}(X)^\heartsuit$ or $(R\text{-mod})^\heartsuit$. We will use homological grading conventions (e.g., "homologically bounded above" = "cohomologically bounded below" = "left t-bounded")

1.7. **Acknowledgements.** The author wishes to warmly thank Jacob Lurie and Boris Tsygan for interesting conversations on the subject of this note, and the CATS 4 organizers for the invitation to speak and for soliciting this note. The author was supported by an NSF Graduate Fellowship and an NSF Postdoctoral Fellowship.

2. Derived completions and Koszul duality

2.1. **Functor of points completion.** We have the following "geometric" approach to completion via the functor of points:

Definition 2.1.1. Suppose that $\pi\colon X \to Y$ is a map of derived algebraic spaces (not necessarily a closed immersion). Then, we take as our functor of points definition of completion $\widehat{Y}$ the following:

$$\widehat{Y}(R) := \left\{ \text{Compatible diagrams} \quad \begin{tikzcd} \operatorname{Spec}^{fp} R_{red} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd} \right\}$$

That is, an R-point of $\widehat{Y}$ consists of an R-point of Y together with a chosen factorization of its restriction to R_{red} through X. The intuition is that $\widehat{Y}$ should be determined by Y and *the deformation theory* of π (not necessarily π itself).

Remark 2.1.2. Consider first the case when π is a closed immersion so that the corresponding map of underlying reduced schemes is a monomorphism. It follows that $\widehat{Y} \to Y$ is a monomorphism – it consists of those R-points of Y that set-theoretically factor through X, with the factorization now being unique if it exists.

There is one further noteworthy special case: If π is a *surjective* closed immersion, then $\widehat{Y} \to Y$ is an isomorphism.

Remark 2.1.3. In the case $Y = \mathrm{pt}$ we get that an R-point of the completion along $X \to \mathrm{pt}$ is just given by $X(R_{red})$ – in other words $\widehat{\mathrm{pt}} = X_{\mathrm{dR}}$. Furthermore, we can describe all other completions in our sense in terms of de Rham space: One has a Cartesian diagram

$$\begin{tikzcd} \widehat{Y} \arrow[r] \arrow[d] & X_{\mathrm{dR}} \arrow[d, "\pi_{\mathrm{dR}}"] \\ Y \arrow[r] & Y_{\mathrm{dR}} \end{tikzcd}$$

.

2.2. **Algebraic completion (and filtration).** This subsection is not strictly needed, but provides interesting motivation.

2.2.1. Suppose given a map $R \to R'$ of *ordinary* (discrete) algebras. In this case, we can take I to be the kerenl of this homomorphism, and consider the I-adic filtration on R / the I-adic completion $\widehat{R}$ of R, and so on. We can ask: what is the "derived" analog of this? There is an obvious complication: the notion of an ideal, and worse of a power of an ideal, is no longer so simple.

It turns out that there *is* a good definition, it just happens to be a little complicated to formulate:

2.2.2. Suppose given a map $R \to R'$ of E_∞-algebras. Even if we might not know what I should be, we *do* know what I/I^2 should be – this should be $\Omega\mathbb{L}_{R'|R}$, a shift of the cotangent complex. This means that we know what each of R/I and R/I^2 should be: R/I should be R'; and, R/I^2 should be the universl square zero extension of R by $\Omega\mathbb{L}_{R'|R}$ in the sense of [L2].

Thinking about the formulation of a "square zero extension" in [L2], one can come up with a generalization:

Definition 2.2.3. Let $\mathbf{CAlg}_{/R'}$ denote the ∞-category of E_∞-algebras with a map to R'. Let

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[drrrr] & P_n \arrow[r] \arrow[drrr] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dr] & P_0 \arrow[d] \\ & & & & & \mathrm{id} \end{tikzcd}$$

be the Goodwillie tower of the identity functor [L2, Chapter 7].

Then, we define the completion

$$\widehat{R} := \varprojlim_n P_n(R)$$

with the filtration gotten from the stages $P_n(R)$.

2.2.4. That is, $\{P_n\}$ is a tower of functors

$$P_n \colon \mathbf{CAlg}_{/R'} \to \mathbf{CAlg}_{/R'}$$

that better and better approximate the identity via functors that can be thought of as "polynomial-of-degree-n near R'." For instance $P_0(R) = R'$, while $P_1(R)$ is the universal square-zero extension, i.e., it fits into a fiber square of E_∞-algebras

$$\begin{tikzcd} P_1(R) \arrow[r] \arrow[d] & R' \arrow[d, "\eta"] \\ R' \arrow[r, "0"'] & R' \oplus \mathbb{L}_{R'|R} \end{tikzcd}$$

where η is the universal derivation.

We can ask: How do we ever compute this thing? At least in characteristic zero, it is possible to do it by using Koszul duality. We will begin by demonstrating this in one of the strangest, and most interesting sounding, case:

2.2.5. Suppose that $k \to R$ is a connective, commutative dg-k-algebra. We will explain the following motto:

The derived de Rham complex $C^\bullet_{\mathrm{dR}}(R)$ is the (derived) formal completion of k along R

We do not wish to go into the details of what the universal property of "formal completion" should be in this setting. One can give a definition using the Goodwillie calculus, and make it suitably functorial by an elaboration of the techniques used in [L2] to define the cotangent complex in terms of the "tangent correspondence." Instead, we'll assume that there's a reasonable notion of completion and figure out what the completion of k along R *should be.*

The idea will be to replace $k \to R$ by a map which looks more like an ordinary closed immersion. First, pick a quasi-free resolution $R \simeq k[x_S]$ with some differential $Q(x_S) = \cdots$. Then, replace k by $k \simeq k[x_S, y_S]$ where $\deg y_S = \deg x_S - 1$ (in homological grading) so that we can declare $Q(x_S) = -y_S + \cdots$. Now our map $k \to R$ has been replaced by a map which, ignoring differentials, just looks like

$$k[x_S][y_S] \longrightarrow k[x_S]$$

and this we know how to compute the completion of: it is just $k[x_S][\![y_S]\!]$. So, $k[x_S][\![y_S]\!]$ equipped with the induced differential $Q(x_S) = -y_S + \cdots$ is our candidate for the completion. But now, a moment's thought verifies that *this* is nothing but the completed de Rham algebra of the quasi-free resolution $k[x_S]$!

The procedure described, by taking a semi-free resolution and then making it look like a surjection can be re-worded in fancier terms: We say that $\mathbb{L}_{R'|R}$ is a Lie co-algebroid in R'-mod, so that we can take the algebroid version of the Chevalley-Eilenberg cochains $C^\bullet_{CE/R}(\mathbb{L}_{R'|R})$. This is a filtered, complete, E_∞ algebra whose associated graded is just the free E_∞-R-algebra $\mathrm{Sym}_R\, \mathbb{L}_{R'|R}$.

Remark 2.2.6. Given a map of commutative algebras, we can always forget that they are commutative. Our abstract definition of completion makes sense for E_n algebras for any n, in particular for associative algebras. General non-sense will tell us that the E_∞ completion is the inverse limit of the E_n completions. But we can ask: Is that all really necessary, or will completion commute with forgetting commutativity?

In general, the answer is *no*: If we consider the map $k \to R$, then the E_1-completion is just k while we have seen that the E_∞-completion is $C^\bullet_{dR}(R)$. However, if $R \to R'$ is a finitely presented map of connective E_∞-algebras which is a surjection on π_0 – i.e., a finitely generated closed immersion – then we can show that completion *does* commute with forgetting down to E_1-algebras.

2.2.7. In both cases, the conclusion of the previous Remark can be proven by explicitly identifying the E_1-compltion along our E_∞ map. There is a similar description in terms of Koszul duality and unravelling it we see that it is the *Adams (or bar-cobar) completion*

$$\operatorname{Tot}\left\{(R')^{\otimes_R(\bullet+1)}\right\} = \operatorname{Tot}\left\{R' \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} R' \otimes_R R' \cdots\right\}$$

More simply the problem is visible at the bottom layer. We have that

$$\mathbb{L}^{E_1}_{R'|R} = \text{the augmentation ideal } I \subset R' \otimes_R R' \qquad \text{and} \qquad \mathbb{L}^{E_\infty}_{R'|R} = I/I^2$$

so that $\mathbb{L}^{E_1}_{R'|R}$ carries a filtration with associated graded pieces built from $\mathbb{L}^{E_\infty}_{R'|R}$. But, in general, it need not be complete with respect to this filtration. If $\operatorname{Spec} R' \to \operatorname{Spec} R$ is a closed immersion, then it *will* be complete.

The case of E_n-completions, $n > 1$, is then amenable to an inductive argument. Since the diagonal is a closed immersion, one can check that the E_∞ completion of a map of connective E_∞-algebras agrees with the E_n-completion for $n \geq 2$. This admits a combinatorial description by taking iterated Adams towers. Finally, combining with the results of Theorem 3.3.2[4] we can simultaneously prove the following two assertions:

Proposition 2.2.8. *Suppose that $R \to R'$ is a finitely presented map of connective E_∞-algebras.*

- *(i) Then the E_n-completions $\widehat{R}^{E_n}$ stabilize for $n \geq 2$. If the map is a closed immersion, then they stabilize for $n \geq 1$.*
- *(ii) The E_∞-completion $\widehat{R}$ agrees with global sections on the "geometric" completion. That is, if we consider the induced map $X = \operatorname{Spec} R' \to Y = \operatorname{Spec} R$ and let $\widehat{Y}$ denote the functor-of-points completion, then $\mathrm{R}\Gamma(\widehat{Y}, \mathcal{O}_{\widehat{Y}}) \simeq \widehat{R}$.*

A warning: The filtrations *do depend on n.*

Again we remark on the intuition: For connective algebras, E_2 is commutative enough so that $\pi_0 R$ is an ordinary commutative ring and "geometry" kicks in. Thus, completions for connective E_n-algebras $n \geq 2$ are geometric with the same functor of points.

Finally, we highlight a way in which this section fits into the philosophy of the rest of this paper: In the proof of Theorem 5.1.3 we will take a Cech nerve, and then complete it level-wise. This is exactly to fix this difference between E_1 and E_∞ completions.

2.3. Splitting the filtration by Adams operations.

2.3.1. In the case of the derived loop space LX, it is well-known that

- $\mathcal{O}_{LX} = \mathbf{HH}_\bullet(\mathcal{O}_X)$ admits a decreasing filtration (as $\mathcal{O}_X$-algebra), whose associated graded is closely related to the cotangent complex $\mathbb{L}_X$ of X;
- This filtration naturally splits rationally, giving rise to a "Hodge decomposition." Furthermore, this splitting can be accomplished by considering the actions of certain *power operations* on $\mathcal{O}_{LX}$.

We will need a variant of this that is both more general and weaker: it is weaker in that it considers only the $\mathcal{O}_X$-module, rather than $\mathcal{O}_X$-algebra, structure); it is more general in that it holds not just for $LX = X^{S^1}$ but X^S for any co-H-space (e.g., every suspension).

Proposition 2.3.2. *Suppose that X is a derived algebraic space in characteristic zero. Then,*

- *(i) A finite pointed space $pt \to S$ gives rise to a map $p \colon X^S \to X$ in derived schemes under X. This gives rise to an $\mathcal{O}_X$-module $p_* \mathcal{O}_{X^S}$ functorially in pointed maps of S. More precisely, the above determines a composite sequence of functors*

  $$\mathbf{Spaces}^{fin}_{*} \xrightarrow{X^S} \mathrm{DerSch}_{X//X} \xrightarrow{p_*} \mathrm{QC}(X)$$

  If S is connected, then X^S is a nil-thickening of X.

---

[4] Strictly speaking, that's only finite type over a field. In the generality we can use the E_∞ algebra analog of [HLP, Corollary 4.7].↑

(ii) There is a decreasing filtration on $p_*\mathcal{O}_{X^S} \in \mathrm{QC}(X)$ functorial in pointed maps of S

$$p_*\mathcal{O}_{X^S} = F^0 \supset F^1 \supset F^2 \supset \cdots$$

whose associated graded pieces can be naturally identified with

$$\mathrm{gr}_i F^\bullet \simeq \mathrm{Sym}^i\left(\mathbb{L}_{X/X^S}\right) \simeq \mathrm{Sym}^i\left(\mathbb{L}_X \otimes_k \widetilde{C}_\bullet(S)\right)$$

where $\widetilde{C}_\bullet(S)$ denotes reduced k-linear chains on S. In other words, the above functor lifts to one in a commutative diagram

$$\mathbf{Spaces}_*^{fin} \xrightarrow{X^S} \mathrm{DerSch}_{X//X} \xrightarrow{p_*} \mathrm{Filt}\,\mathrm{QC}(X)$$

(iii) Suppose that S admits a co-H-space structure as pointed space (e.g., S is the suspension of a pointed space). Then, the filtration of (ii) splits naturally and there is an equivalence of $\mathcal{O}_X$-modules

$$p_*\mathcal{O}_{X^S} \xrightarrow{\sim} \mathrm{Sym}_{\mathcal{O}_X}\left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right)$$

Indication of Proof.

(i) The only claim that we must verify is that X^S is a nil-thickening of X when S is connected (and non-empty, since it is pointed!). The claim is local on X, so that we may suppose $X = \operatorname{Spec} R$ and $X^S = \operatorname{Spec} S \otimes R$. Now it is enough to observe that$\pi_0(S \otimes R) = \pi_(S) \otimes \pi_0 R$ where the first (resp., second) use of $\otimes$ denote the tensoring of derived rings (resp., rings) over spaces (resp., sets).

(ii) This is the filtration corresponding to the Goodwillie Tower of the functor in (i). The identification of the layers in the tower ("Goodwillie derivatives") can be deduced as follows: First note that the first functor $S \mapsto X^S$ is excisive, so that the chain rule implies that the Goodwillie derivatives are determined by that of the forgetful functor from augmented R-algebras to R-modules. It is well-known that the first Goodwillie derivative of this forgetful functor is the cotangent complex $\mathbb{L}_R$ (e.g., this is the *definition* of $\mathbb{L}_R$ in [L2]), and that the higher derivatives are given by taking smash products and symmetric co-invariants from the first.[5] For an exposition of a similar topic see [HH].

For the reader's convenience, we also recall (again) a explicit description of the filtration in terms of Koszul duality: First recall that by definition

$$\begin{aligned}\mathrm{Map}_{\mathrm{DerSch}}(\operatorname{Spec} A, (\operatorname{Spec} R)^S) &= \mathrm{Map}_{\mathbf{sSet}}(S, \mathrm{Map}_{\mathrm{DerSch}}(\operatorname{Spec} A, \operatorname{Spec} R)) \\ &= \mathrm{Map}_{\mathbf{sSet}}(S, \mathrm{Map}_{\mathbf{DRng}}(R, A)) \\ &= \mathrm{Map}_{\mathbf{DRng}}(R, A^S) \\ &= \mathrm{Map}_{\mathbf{DRng}}(R \otimes S, A)\end{aligned}$$

where $A^S = A \otimes_k \Omega^\bullet(S)$ with $\Omega^\bullet(S)$ denoting algebraic de Rham forms on the simplicial set S. This is the usual description of the co-tensoring of **DRng** over **sSet**: since S is a finite simplicial set, $A \otimes_k \Omega^\bullet(S)$ is a model for the mapping spectrum $\mathrm{RHom}(S, H(A))$. While the co-tensoring by spaces ($A \mapsto A^S$) was immediate from the viewpoint of commutative algebras, it turns out that the tensoring ($R \mapsto R \otimes S$) is better seen from the Koszul dual viewpoint of Lie co-algebras:

Recall that $R \otimes S$ is an augmented commutative R-algebra. The relative cotangent complex

$$\mathbb{L}_{R/R\otimes S} = \mathbb{L}_{R\otimes S/R}\big|_R[+1]$$

carries an R-linear Lie co-algebra structure, and there is a Koszul duality map of R-algebras

$$\mathrm{KD}\colon C_\bullet^{\mathrm{coLie}/R}(\mathbb{L}_{R/R\otimes S}) \longrightarrow R \otimes S$$

Furthermore, since formation of cotangent complexes preserves colimits of commutative algebras one may identify the underlying complex

$$\mathbb{L}_{R/R\otimes S} \simeq \mathbb{L}_R[+1] \otimes_k \widetilde{C}_\bullet(S)$$

---

[5] This latter fact, for which we can't find a precise reference, can be proved by combining a general description of the layers in terms of cross effects, the fact that coproduct in commutative algebras is the smash product, and the computation of the linear piece.↑

One can then check that KD induces an equivalence on cotangent complexes and on reduced parts (recall that S is connected), and is consequently an equivalence. Finally, the claim of the Proposition follows from the above identification of $\mathbb{L}_{R/R\otimes S}$ by noting that the underlying R-module of $C^{\mathrm{coLie}/R}(\mathscr{L})$ is filtered with associated graded $\mathrm{Sym}_R(\mathscr{L}[-1])$: It must be filtered because the Chevalley-Eilenberg differential includes an "extraneous" term, and passing to associated gradeds for the filtrations turns off this term.

(iii) This is a general statement on splitting of the Goodwillie tower at a co-H-space. [BM, Theorem 1.1] □

Remark 2.3.3. We will talk about Ind-coherent complexes below, but let us note that the previous Proposition essentially explains why they will be convenient for us: If X is almost of finite presentation, then $\mathbb{L}_X$ will be almost perfect. Using the Postnikov filtration, we view $\mathbb{L}_X$ as an inverse limit of bounded coherent modules on X. Combined with the filtration in (ii) this tells us that that $p_*\mathcal{O}_{X^S}$ has extra structure beyond that of an object of $\mathrm{QC}(X)$ – namely, it should be considered as a *Pro-coherent* complex.

Since we prefer algebras to coalgebras, we tend to prefer Ind-constructions to Pro-constructions. Now taking the Grothendieck dual of the above we obtain that $p_*\omega_{X^S}$ should be naturally considered as an Ind-coherent complex.

Remark 2.3.4. Surprisingly, the characteristic zero assumption in the previous Proposition is not *entirely* essential. For (i) and (ii) it is not needed at all – while the Koszul duality description, as written, may have depended on characteristic zero the result as stated is true over the sphere spectrum. For (iii), it is enough to assume that the norm map is an equivalence for Σ_n-modules – this is true in characteristic zero, but also in e.g., $K(n)$-local homotopy theory.

3. IND-COHERENT COMPLEXES

3.1. The formal structure: the functors, base change, etc.

We refer the reader to [P1, G1] for a more detailed introduction, but in the meantime do want to give some flavor:

Definition 3.1.1. If X is a quasi-compact and quasi-separated algebraic space of finite type over k, we have $\mathrm{DCoh}(X) \subset \mathrm{QC}(X)$ the full subcategory of those complexes $\mathscr{F}$ that have bounded coherent homology sheaves. Furthermore, we set

$$\mathrm{QC}^!(X) := \mathrm{Ind}(\mathrm{DCoh}(X)).$$

If $f\colon X \to Y$ is a morphism of such, we have two natural functors:

$$f^!\colon \mathrm{QC}^!(Y) \to \mathrm{QC}^!(X) \qquad \text{and} \qquad f_*\colon \mathrm{QC}^!(X) \to \mathrm{QC}^!(Y).$$

(As usual, ! is read as "shriek.") This gives rise to a distinguished objects: $\omega_X := \pi^! k$ where $\pi\colon X \to \operatorname{Spec} k$ is the projection.

Remark 3.1.2. The notation $\mathrm{QC}^!$ is motivated by the following. A better definition would have been to make the above definition, in terms of $\mathrm{Ind}(\mathrm{DCoh}(-))$, only for *affine* schemes – and then to define $\mathrm{QC}^!(X)$ for general X by Kan extension from affines. And indeed, in general this is what we must do. It then becomes a Theorem that $\mathrm{QC}^!$ is compactly generated by DCoh in many favorable cases – these include quasi-compact and quasi-separated algebraic spaces and geometric stacks in characteristic zero.

Remark 3.1.3. In the notation of the next section, $\mathrm{QC}^!(X) = \mathscr{R}(\mathrm{QC}(X))$ is the regularization.

Definition 3.1.4. $\mathrm{QC}^!(X)$ is a module over the symmetric monoidal structure on $\mathrm{QC}(X)$: that is, the symmetric monoidal category $\mathrm{Perf}(X)$ acts on $\mathrm{DCoh}(X)$ and similarly upon passing to Ind-completions. We denote this action simply by "$\otimes''$.

There is also a symmetric monoidal structure on $\mathrm{QC}^!(X)$ itself, coming from the exterior product and the shriek pullback along the diagonal

$$\mathscr{F} \overset{!}{\otimes} \mathscr{G} := \Delta^!(\mathscr{F} \boxtimes \mathscr{G})$$

having monoidal unit ω_X.

The functor

$$\omega_X \otimes -\colon \mathrm{QC}(X) \longrightarrow \mathrm{QC}^!(X)$$

is symmetric monoidal.

The actual relations satisfied by the functors above are complicated, and best expressed in terms of a suitable $(\infty, 2)$-category of correspondences. This is discussed in [G1, Section 5]. We content ourselves with a sketchier picture:

3.1.5. If f is proper, then f_* is left adjoint to $f^!$. If f is étale, then $f^!$ is left adjoint to f_*. Given a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"'] & Y \end{tikzcd}$$

there is a base change equivalence

$$p_*(f')^! \simeq f^! q_*$$

3.2. **Shriek integral transforms.** One of the basic facts about $\mathrm{QC}^!$ is that absolute tensor / functor theorems still hold. (They do not, in general, over a base!)

Theorem 3.2.1 ([P1, Appendix B])**.** *The shriek Fourier-Mukai transform provides an equivalence*

$$\Phi^!_{(-)} \colon \mathrm{QC}^!(X \times Y) \xrightarrow{\sim} \mathrm{Fun}^L(\mathrm{QC}^!(X), \mathrm{QC}^!(Y))$$

where

$$\Phi^!_{\mathscr{K}}(\mathscr{F}) = (p_2)_* \left((p_1)^!(\mathscr{F}) \overset{!}{\otimes} \mathscr{K} \right)$$

In the case $X = Y$, this is a monoidal equivalence – where $\mathrm{QC}^!(X^2)$ is equipped with the the monoidal structure of shriek-convolution, and monoidal unit $\Delta_* \omega_X$.

3.3. **Descent (and completion) results.** It turns out that $\mathrm{QC}^!$ has quite a lot of descent, and is in many ways more convenient than QC when working with formal completions.

Theorem 3.3.1 ([P1, Appendix A])**.** *Suppose that $\pi \colon X \to Y$ is an h-cover (e.g., an fppf cover, or a proper surjection). Let* $\mathrm{Cech}(\pi)$ *denote the Cech nerve of π as an augmented simplicial diagram. Then, the shriek pullback induces an equivalence*

$$\mathrm{QC}^!(Y) \xrightarrow{\sim} \mathrm{Tot}\left\{ \mathrm{QC}^!(\mathrm{Cech}(\pi)) \right\}$$

In fact, even more is true. If π is a closed immersion but *not* surjective, we can still say what happens:

Theorem 3.3.2 ([P1, Section 5])**.** *Suppose that $\pi \colon X \to Y$ is proper, $|Z|$ its set-theoretic image, and $\widehat{X}$ the completion of X along $|Z|$. Let* $\mathrm{Cech}(\pi)$ *denote the Cech nerve of π. Then, the shriek pullback induces equivalences*

$$\mathrm{QC}^!_Z(X) \xrightarrow{\sim} \mathrm{QC}^!(\widehat{X}) \xrightarrow{\sim} \mathrm{Tot}\left\{ \mathrm{QC}^!(\mathrm{Cech}(\pi)) \right\}$$

Remark 3.3.3. In [P1, Section 5] this is only stated with π a closed immersion. In fact, the same proof holds.

Remark 3.3.4. The above Theorem also holds – with a similar proof – in case the map π is not representable, but merely *ind-finite* (i.e., a suitable colimit of finite morphisms). The proto-typical example of this is the projection $\pi \colon X \to \widehat{\mathrm{pt}} = X_{\mathrm{dR}}$. In this example, the terms of the Cech nerve are nothing but the terms of the Cech nerve for $X \to \mathrm{pt}$ completed along their diagonals

$$\begin{tikzcd}[column sep=large] X_{\mathrm{dR}} \arrow[r] & X \arrow[l, shift left=1.2ex] \arrow[l, shift right=1.2ex] \arrow[r, shift left=0.6ex] \arrow[r, shift right=0.6ex] & \widehat{X^2} \arrow[l] \arrow[l, shift left=1.2ex] \arrow[l, shift right=1.2ex] \arrow[r] \arrow[r, shift left=1.2ex] \arrow[r, shift right=1.2ex] & \cdots \arrow[l, shift left=0.6ex] \arrow[l, shift right=0.6ex] \arrow[l, shift left=1.8ex] \arrow[l, shift right=1.8ex] \end{tikzcd}$$

and this is the usual description of $\mathrm{QC}^!(X_{\mathrm{dR}})$ in terms of crystals on the infinitesimal groupoid.

3.4. **Crystals.** In this paper we take $\mathrm{QC}^!(X_{\mathrm{dR}})$ as a definition of "D-module," $\mathrm{R}\Gamma(X_{\mathrm{dR}}, \omega_{X_{\mathrm{dR}}})$ as a definition of "(de Rham) Borel-Moore chains," etc. The curious reader is directed to [GR3] for a verification that this is compatible with other classical definitions.

4. REGULARIZING t-STRUCTURES

The goal of this section is to recall some constructions having to do with ∞-categories with t-structure. Constructions similar to what we call $\mathscr{R}$ appear in [FG, Section 22] and ideas similar to those exposed here have also been worked out by J. Lurie in unpublished work. The present exposition, which appears also as an Appendix in [BZNP], is a survey form of the Appendix to the author's coming pre-print [P2].

4.1. Completions of t-structures.

For the reader's convenience, we recall several convenient conditions and constructions with t-structures from [L2]:

Definition 4.1.1. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure. We say that the t-structure is *compatible with filtered colimits* if $\mathcal{C}$ has all filtered colimits and $\mathcal{C}_{<0}$ is closed under filtered colimits in $\mathcal{C}$.

Definition 4.1.2. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure.

- We say that the t-structure is (weakly) *left complete* if the natural map
$$\mathscr{F} \longrightarrow \varprojlim_n \tau_{<n}\mathscr{F}$$
is an equivalence for all $\mathscr{F} \in \mathcal{C}$ (in particular, the inverse limit is required to exist).

  We say that it is left complete if furthermore every tower in $\varprojlim \mathcal{C}_{<n}$ comes from an object of $\mathcal{C}$.
- We say that the t-structure is (weakly) *right complete* if the natural map
$$\varinjlim_n \tau_{\geq n}\mathscr{F} \longrightarrow \mathscr{F}$$
is an equivalence for all $\mathscr{F} \in \mathcal{C}$ (in particular, the direct limit is required to exist).

  We say that it is right complete if furthermore every diagram of objects in $\varprojlim \mathcal{C}_{>-n}$ comes from an object of $\mathcal{C}$.

Remark 4.1.3. The previous definition is of course formally symmetric: a t-structure on $\mathcal{C}$ is left complete iff the opposite t-structure on $\mathcal{C}^{\mathrm{op}}$ is right complete. In practice there is however a substantial asymmetry: We are generally interested in presentable categories, and the opposite of a presentable category is almost never presentable. More practically, the categories that arise in algebraic geometry – at least for our purposes – tend to be right-complete, but some interesting categories fail to be left-complete.

Remark 4.1.4. By [L2, 1.2.1.19], this distinction between the "(weakly)" and not variants disappears for the notion of left complete (resp., right complete) provided that $\mathcal{C}$ has countable products (resp., coproducts), and that countable products are right t-exact (resp., coproducts are left t-exact) up to a finite shift.

In particular, if $\mathcal{C}$ is presentable and the t-structure compatible with filtered colimits then "weakly left complete" coincides with "left complete."

Example 4.1.5. Suppose $A \in \mathbf{Alg}(k\text{-mod})$. When can we equip A-mod with a (nice) t-structure? There are two notable cases:

- If A is connective (i.e., $A \in (k\text{-mod})_{\geq 0}$), then A-mod carries an accessible t-structure which is left and right complete. It is defined by letting the conservative forgetful functor $\theta\colon A\text{-mod} \to k\text{-mod}$ be t-exact, i.e., $(A\text{-mod})_{>0} = \theta^{-1}(k\text{-mod}_{>0})$ and $(A\text{-mod})_{<0} = \theta^{-1}(k\text{-mod}_{<0})$. The heart of this t-structure identifies with ordinary (discrete) modules over $\pi_0(A)$.
- On the hand if $A \in (k\text{-mod})_{\leq 0}$ then A-mod carries an accesible t-structure which is right complete (but not left complete). It is defined by declaring $(A\text{-mod})_{<0} = \theta^{-1}(k\text{-mod}_{<0})$, while the connective objects are harder to recognize – $(A\text{-mod})_{\geq 0}$ is the smallest full subcategory that contains the objects $A \otimes M$ for $M \in (k\text{-mod})_{\geq 0}$ iand is closed under colimits and extensions. It follows that θ is left t-exact, while the free module functor $A \otimes -$ is t-exact. The heart of this t-structure can also be identified with ordinary (discrete) modules over $\pi_0(A)$.

For an example of this second case, consider $A = H^*(BS^1, k) \simeq k[\![u]\!]$ where u is in homological degree -2. In this case, an A-module M is co-connective if and only if its underlying module is co-connective; and an A-module M is connective if and only if $k \otimes_A M$ is connective. In particular:

- $A \in (A\text{-mod})^{\heartsuit}$;
- $k[+n] \in (A\text{-mod})_{\geq 0}$ if and only if $n \geq 1$ (k itself is not connective!).

- $k((\beta))$ is a non-zero object which is in $(A\text{-mod})_{>n}$ for all n, so that the t-structure is not left complete.
- The map
$$M \longrightarrow \mathrm{RHom}_{\mathrm{RHom}_A(k,k)}(k, k \otimes_A M)$$
identifies the right-hand term with $\tau_{\le 0}M$. In other words, we are transferring over the t-structure on modules over the connective algebra $\mathrm{RHom}_A(k,k) = H_*(S^1, k)$.

4.2. Coherent and Noetherian t-structures.

Assuming some extra "finiteness" conditions on the t-structure, one has extra operations of *regularization* available in addition to *completion*.

Lemma 4.2.1. *Suppose $\mathcal{C}$ is a stable ∞-category with t-structure that is compatible with filtered colimits. For $\mathscr{F} \in \mathcal{C}$, the following conditions are equivalent*

(i) $\tau_{<n}\mathscr{F} \in \mathcal{C}_{<n}$ *is compact for all* $n \in \mathbb{Z}$;

(ii) $\mathrm{Map}_{\mathcal{C}}(\mathscr{F}, -)$ *commutes with filtered colimits in* $\mathcal{C}_{<n}$ *for all* $n \in \mathbb{Z}$ *("commutes with uniformly bounded above colimits");*

(iii) $\mathrm{RHom}_{\mathcal{C}}(\mathscr{F}, -)$ *commutes with filtered colimits in* $\mathcal{C}_{<n}$ *for all* $n \in \mathbb{Z}$.

Furthermore,

- *Suppose in addition that $\mathscr{F}$ is assumed bounded above: $\mathscr{F} \in \mathcal{C}_{<n}$. Then, the above are equivalent to: $\mathscr{F} \in (\mathcal{C}_{<n})^c$ and its image under the inclusion $i_{<n}\colon \mathcal{C}_{<n} \to \mathcal{C}_{<m}$ is compact for all $m \ge n$;*
- *Suppose that $\mathcal{C}$ is right complete. If $\mathscr{F}$ is bounded above and satisfies the above equivalent conditions, then it is also bounded below.*

Definition 4.2.2. Say that $\mathscr{F} \in \mathcal{C}$ is *almost compact* if $\mathscr{F}$ satisfies the equivalent conditions of the previous Lemma. Say that $\mathscr{F} \in \mathcal{C}$ is *coherent* if

(i) $\mathscr{F}$ is bounded above, i.e., $\mathscr{F} \in \mathcal{C}_{<n}$ for some n.

(ii) $\mathscr{F}$ satisfies the equivalent conditions of the previous Lemma.

(If $\mathcal{C}$ is right complete, then any such $\mathscr{F}$ is also bounded below by the previous Lemma.)

Define the full subcategory $\widehat{\mathrm{Coh}}_+(\mathcal{C}) \subset \mathcal{C}$ *(resp.,* $\mathrm{Coh}(\mathcal{C}) \subset \mathcal{C}$*)* to consist of all $\mathscr{F} \in \mathcal{C}$ that are almost compact (resp., coherent).[6]

Remark 4.2.3. Characterization (iii) of the previous Lemma makes clear that $\widehat{\mathrm{Coh}}_+(\mathcal{C})$ and $\mathrm{Coh}(\mathcal{C})$ are stable subcategories. Notice that, in general, the the t-structure need not restrict to these subcategories.

We can impose the following more stringent conditions to eliminate this issue:

Lemma 4.2.4. *Suppose $\mathcal{C}$ is a stable ∞-category with t-structure that is compatible with filtered colimits. Then, the following conditions are equivalent:*

(i) The t-structure on $\mathcal{C}$ restricts to one on $\widehat{\mathrm{Coh}}_+(\mathcal{C})$;

(ii) The truncation functors on $\mathcal{C}$ preserves $\widehat{\mathrm{Coh}}_+(\mathcal{C})$.

(iii) The inclusion $i_{<0}\colon \mathcal{C}_{<0} \to \mathcal{C}_{<1}$ preserves compact objects;

(iv) The loops functor $\Omega\colon \mathcal{C}_{<0} \to \mathcal{C}_{<0}$ preserves compact objects;

In this case, $\widehat{\mathrm{Coh}}_+(\mathcal{C})^\heartsuit = \mathrm{Coh}(\mathcal{C})^\heartsuit = (\mathcal{C}^\heartsuit)^c$.

These imply – and in case $\mathcal{C}$ is right complete, are equivalent to –

(v) The subcategory of compact-objects in the heart $(\mathcal{C}^\heartsuit)^c \subset \mathcal{C}^\heartsuit$ is abelian;

Under the above hypotheses, we have:

Lemma 4.2.5. *Suppose $\mathcal{C}$ is a stable ∞-category with t-structure that is compatible with filtered colimits, right complete, and satisfies the equivalent conditions of Lemma 4.2.4. Then:*

- *$\mathrm{Coh}(\mathcal{C})^\heartsuit = \mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^\heartsuit = (\mathcal{C}^\heartsuit)^c$ consists precisely of the compact (or "finitely presented") objects of $\mathcal{C}^\heartsuit$.*
- *$\mathscr{F} \in \widehat{\mathrm{Coh}}_+(\mathcal{C})$ if and only if $H_n\mathscr{F} \in \mathrm{Coh}(\mathcal{C})^\heartsuit \subset \mathcal{C}^\heartsuit$ and $H_n\mathscr{F} = 0$ for $n \ll 0$;*
- *$\mathscr{F} \in \mathrm{Coh}(\mathcal{C})$ if and only if $H_n\mathscr{F} \in \mathrm{Coh}(\mathcal{C})^\heartsuit \subset \mathcal{C}^\heartsuit$ and $H_n\mathscr{F} = 0$ for all but finitely many n.*

---

[6] This notation is potentially confusing, but fortunately will not be used much in general: $\widehat{\mathrm{Coh}}_+(\mathcal{C})$ need not be the left t-completion of $\mathrm{Coh}(\mathcal{C})$ in general.↑

Finally, we come to a strengthening of the above:

Lemma 4.2.6. *Suppose that $\mathcal{C}$ is a stable ∞-category with t-structure that is compatible with filtered colimits, that is right complete and that satisfies any of the equivalent conditions of Lemma 4.2.4.*

Then, the following conditions are equivalent:

- *(i)* $\mathcal{C}_{<0}$ *is compactly-generated as ∞-category (in particular, presentable);*
- *(ii)* $\mathcal{C}^{\heartsuit}$ *is compactly-generated as ordinary category;*
- *(iii)* $\mathcal{C}^{\heartsuit}$ *is a locally coherent abelian category. (Recall this means that the compact objects form an abelian category, and that $\mathcal{C}^{\heartsuit}$ is compactly generated. In particular, it is Grothendieck.)*

This brings us to the following definition (which the previous Lemmas give various equivalent formulations and consequences of):

Definition 4.2.7. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure. We say that the t-structure is *coherent* if the following conditions are satisfied:

- The t-structure is compatible with filtered colimits;
- The t-structure is right complete;
- $\mathcal{C}^{\heartsuit}$ is a locally coherent abelian category.

4.3. **Regular and complete t-structures.**

Definition 4.3.1. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure compatible with filtered colimits. We have seen that $\mathcal{C}_{<0} \to \mathcal{C}_{<1}$, etc., preserves filtered colimits. We say that the t-structure is (left) *regular* if it is coherent and the natural map

$$\mathrm{Fun}^{\text{filtered colimits}}(\mathcal{C}, \mathcal{D}) = \varprojlim_n \mathrm{Fun}^{\text{filtered colimits}}((\mathcal{C}_{<n}, i_{<n}), \mathcal{D})$$

is an equivalence for every category $\mathcal{D}$ admiting filtered colimits.

Proposition 4.3.2. *Suppose that $\mathcal{C}$ is* coherent. *Then, $\mathcal{C}$ is regular if and only if $\mathcal{C}$ is compactly-generated by* $\mathrm{Coh}(\mathcal{C})$.

There is a universal regular ∞-category with t-structure mapping to $\mathcal{C}$, and it is given by the formula

$$\mathscr{R}(\mathcal{C}) \overset{def}{=} \mathrm{Ind}\left(\mathrm{Coh}(\mathcal{C})\right) \longrightarrow \mathcal{C}$$

The functor $\mathscr{R}(\mathcal{C}) \to \mathcal{C}$ preserves colimits, is t-exact, and the induced functor $\mathscr{R}(\mathcal{C})_{<0} \to \mathcal{C}_{<0}$ is an equivalence. The t-structure on $\mathscr{R}(\mathcal{C})$ is also coherent.

Proof. By hypothesis, $\mathcal{C}_{<0}$ is compactly generated with compact objects $\mathrm{Coh}(\mathcal{C})_{<0}$. The functors $\mathcal{C}_{<0} \to \mathcal{C}_{<1}$ preserve both filtered colimits and compact objects, so we see that

$$\mathrm{colim}_n^{\text{filtered colimits}} \mathcal{C}_{<n} = \mathrm{Ind}\left(\mathrm{colim}_n \mathrm{Coh}(\mathcal{C})_{<n}\right) = \mathrm{Ind}\left(\mathrm{Coh}(\mathcal{C})\right)$$

In particular, the first colimit exists: This is the assertion that there is an ∞-category with the correct universal property; and it is given by the desired formula.

Notice that the functor $\mathscr{R}(\mathcal{C}) \to \mathcal{C}$ preserves filtered colimits by construction, and finite colimits on compact objects by inspection, so that it preserves colimits. Since both t-structures are compatible with filtered colimits, and since

$$\mathscr{R}(\mathcal{C})_{<0} = \mathrm{Ind}(\mathrm{Coh}(\mathcal{C})_{<0}) \qquad \mathscr{R}(\mathcal{C})_{>0} = \mathrm{Ind}(\mathrm{Coh}(\mathcal{C})_{>0})$$

by construction, we see that this functor is t-exact. It is evident that it induces an equivalence on co-connective objects.

Let us verify that $\mathscr{R}(\mathcal{C})$ is coherent: The t-structure is compatible with filtered colimits, as $\mathscr{R}(\mathcal{C})_{<0} \to \mathscr{R}(\mathcal{C})$ preserves filtered colimits by construction. It is right complete and satisfies the extra coherent condition, since these both depend only on $\mathscr{R}(\mathcal{C})_{<0} \simeq \mathcal{C}_{<0}$. ☐

Definition 4.3.3. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure. We say that the t-structure is (left) *complete* if the natural functor

$$\mathcal{C} \to \varprojlim_n (\mathcal{C}_{<n}, \tau_{<n})$$

is an equivalence.[7]

Proposition 4.3.4. *Suppose that $\mathcal{C}$ is a stable ∞-category with t-structure. Then, there is a universal complete ∞-category with t-structure mapping to $\mathcal{C}$, and it is given by the formula*

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}} = \varprojlim_n \mathcal{C}_{<n}$$

This functor is t-exact, and the induced functor $\mathcal{C}_{<0} \to \widehat{\mathcal{C}}_{<0}$ is an equivalence.

If $\mathcal{C}$ is coherent, then the t-structure on $\widehat{\mathcal{C}}$ is also coherent.

Proof. See [L2, §1.2.1] for everything but the last sentence.

Let us verify that $\widehat{\mathcal{C}}$ is coherent if $\mathcal{C}$ is: The t-structure is compatible with filtered colimits since each functor in the inverse limit is so, and the other properties depend only on $\widehat{\mathcal{C}}_{<0} \simeq \mathcal{C}_{<0}$. ☐

The point of making these definitions is the following:

Definition 4.3.5.

(i) Let $\mathbf{Coh}_t$ denote the ∞-category whose objects are ∞-categories $\mathcal{C}$ with coherent t-structure; whose 1-morphisms are colimit preserving and t-exact functors; and whose higher morphisms are as in $\mathbf{Cat}_\infty$.

(ii) Let $\mathbf{Reg}_t \subset \mathbf{Coh}_t$ denote the subcategory whose objects are ∞-categories $\mathcal{C}$ with *regular* t-structure.

(iii) Let $\mathbf{Cplt}_t \subset \mathbf{Coh}_t$ denote the subcategory whose objects are ∞-categories $\mathcal{C}$ with *complete* (and coherent) t-structure.

Theorem 4.3.6. *The composites*

$$\mathcal{C} \mapsto \mathscr{R}(\mathcal{C})\colon \mathbf{Cplt}_t \hookrightarrow \mathbf{Coh}_t \longrightarrow \mathbf{Reg}_t$$

$$\mathcal{C} \mapsto \widehat{\mathcal{C}}\colon \mathbf{Reg}_t \hookrightarrow \mathbf{Coh}_t \longrightarrow \mathbf{Cplt}_t$$

are inverse equivalences of ∞-categories. (i.e., this is an example of a localization and a co-localization being the same)

Proof. We can show that the two functors are adjoint. It is enough to check that the unit and co-unit are equivalences. For instance if $\mathcal{C} \in \mathbf{Reg}_t$ then we must check that

$$\mathscr{R}(\widehat{\mathcal{C}}) \longrightarrow \mathcal{C}$$

is an equivalence. Since both are regular and the functor is left t-exact and preserves filtered colimits, it is enough to note that it is an equivalence on co-connective objects, which we have seen. The argument for the other adjoint is similar. ☐

4.4. Quasi-coherent and Ind-coherent complexes.

Proposition 4.4.1. *Suppose that X is a geometric stack; or, that X is a quasi-compact and quasi-separated algebraic space. Then, $\mathrm{QC}(X)$ is a stable presentable ∞-category with accessible t-structure. This t-structure is both left and right complete. If X is Noetherian, then $\mathrm{QC}(X)$ is coherent.*

Proof. See [L1] for the first two sentences of the Proposition. For the third, it is a classical statement that every object of $\mathrm{QC}(X)^\heartsuit$ is a filtered colimit of its coherent subobjects. ☐

Example 4.4.2. Suppose A is a Noetherian ring. Then, A-mod carries a t-structure that is both left and right complete in the strong sense. Meanwhile, the full subcategory $\mathrm{DCoh}\, A \subset A$-mod carries a t-structure which is both left and right bounded. In particular, it is weakly left and right complete, though not strongly so. This fully faithful exact embedding into a left (resp., right) complete category identifies the left (resp., right) completion of $\mathrm{DCoh}\, A$ with full subcategories of A-mod:

- The left completion of $\mathrm{DCoh}\, A$ identifies with $\widehat{\mathrm{DCoh}_+}\, A$, the full-subcategory of modules M with $H_i(M)$ coherent over $H_0(A)$ for all i and vanishing for $i \ll 0$;

---

[7] This is just a reformulation of the earlier definition!↑

- The right completion of $\mathrm{DCoh}\, A$ identifies with $\widehat{\mathrm{DCoh}_-}\, A$, the full-subcategory of modules M with $H_i(M)$ coherent over $H_0(A)$ for all i and vanishing for $i \gg 0$;
- The left completion of the right completion (equivalently the other way around) of $\mathrm{DCoh}\, A$ identifies with $\widehat{\mathrm{DCoh}_\pm}\, A$, the full sub-category of modules M with $H_i(M)$ coherent over $H_0(A)$ for all i.

This provides an "application" of the formal symmetry of the definitions. Suppose that $\omega \in A$-mod is a dualizing complex. This means that ω has homologically bounded above coherent homology, finite injective dimension, and the natural map $A \to \mathrm{RHom}_A(\omega, \omega)$ is an equivalence. It follows that the induced duality functor

$$\mathbb{D} = \mathrm{RHom}_A(-, \omega) \colon \mathrm{DCoh}\, A^{op} \longrightarrow \mathrm{DCoh}\, A$$

is an equivalence and that it is left and right t-exact up to finite shifts (where the opposite category gets the opposite t-structure). By formal nonsense, it induces an equivalence on left completion of right completions

$$\mathbb{D} \colon \left(\widehat{\mathrm{DCoh}_\pm}\, A\right)^{op} \simeq \widehat{\mathrm{DCoh}_\pm}\, A$$

Proposition 4.4.3. *Suppose that X is a geometric stack of finite type over a characteristic zero field k; or, that X is a quasi-compact and quasi-separated algebraic space of finite type over a perfect field k. Then, $\mathrm{QC}^!(X)$ is a stable presentable ∞-category with accessible t-structure. This t-structure is coherent and regular.*

Furthermore, the natural map

$$\mathrm{QC}^!(X) \to \mathrm{QC}(X)$$

realizes $\mathrm{QC}^!(X)$ as the regularization of $\mathrm{QC}(X)$, and $\mathrm{QC}(X)$ as the completion of $\mathrm{QC}^!(X)$.

Proof. For the stacky case see [DG]: One uses a finite-length stratification by global quotient stacks to show that X has finite cohomological dimension (this is where one uses characteristic zero); from this, we deduce that $\mathrm{QC}^!(X)^c = \mathrm{DCoh}(X)$. Then, one uses the stratification to show that $\mathrm{QC}^!(X)^\heartsuit$ generates, reducing to the statement about ordinary quasi-coherent sheaves being unions of their coherent subsheaves.

For the algebraic space case, see [BZNP] – in op.cit. it is shown that $\mathrm{DCoh}(X)$ compactly generates $\mathrm{QC}^!(X)$. □

Finally, we have the variants of this for X_{dR}:

Proposition 4.4.4. *Suppose that X is a quasi-compact and quasi-separated algebraic space of finite type over a characteristic zero field k. Then, $\mathrm{QC}^!(X_{\mathrm{dR}})$ is left complete, right complete, and regular.*

Proof. First notice that $\mathrm{QC}^!(X_{\mathrm{dR}})$ is compactly generated rather more generally by [DG, Theorem 8.1.1]. (Though to get the case of algebraic spaces we need to run a variant of that argument, using the existence of scallop decompositions as in [L1]. c.f., the proof that DCoh compactly-generates $\mathrm{QC}^!$ in this case given in [BZNP].)

It remains to prove that the t-structure is left complete, right complete, compatible with filtered colimits, and that the compact objects are preserved by the t-structure. Notice that all of these claims are étale local since $f^! = f^*$ is t-exact for f an étale map. Thus, we may suppose that X is an affine scheme.

Hence we may pick a closed immersion $X \to M$ with M smooth and affine, so that

$$\mathrm{QC}^!(X_{\mathrm{dR}}) \simeq \mathrm{QC}^!_X(M_{\mathrm{dR}}) = \mathrm{fib}\left\{\mathrm{QC}^!(M_{\mathrm{dR}}) \to \mathrm{QC}^!(U_{\mathrm{dR}})\right\}.$$

where $U = M \setminus X$. Then, formal non-sense shows that

$$\mathrm{QC}^!(X_{\mathrm{dR}})^c = \mathrm{QC}^!_X(M_{\mathrm{dR}}) \cap \mathrm{QC}^!(M_{\mathrm{dR}})^c$$

It is then straightforward to reduce to the analogous statements with X replaced by M. But now $\mathrm{QC}^!(M_{\mathrm{dR}})$ identifies with right D_M-modules, and D_M is a regular Noetherian ring. This completes the proof. □

Remark 4.4.5. Note that if X is an Artin stack, even a reasonable one, then $\mathrm{QC}^!(X_{\mathrm{dR}})$ need not be regular. By [DG, Theorem 8.1.1, 6.2.3] it will be compactly generated and (left) complete. But op.cit. Remark 8.1.2 observes that the compact objects will not, generally, be preserved by the truncation functors.

4.5. **Finite simplicial group actions.** The Tate construction is complicated because it mixes a colimit and a limit. In particular, standard techniques of "algebra" – e.g., module categories – are much better suited to colimits than to limits. The key simplifying observation for us is that for $G = \mathrm{SO}(2)$ the formation of $\mathrm{SO}(2)$ invariants behaves as if it were a colimit *when restricted to homologically bounded above complexes*:

4.5.1. When dealing with explicit chain complexes, it is easy to make sense of this: If we identify $(k\text{-mod})^{\mathrm{SO}(2)} \simeq H_*(S^1, k)\text{-mod} \simeq k[B]\text{-mod}$ where B is in homological degree $+1$,[8] then the invariants functor is given on dg-$k[B]$ modules by

$$(V, d) \mapsto (V[\![u]\!], d + Bu)$$

and if V is homologically bounded above then the map

$$(V[u], d + Bu) \longrightarrow (V[\![u]\!], d + Bu)$$

is actually a bijection of underlying graded modules. The left hand-side looks a lot like a colimit (e.g., it preserves filtered colimits, is compatible with tensor in a precise sense, etc.)!

We may codify this observation in the following Lemma:

Lemma 4.5.2. *Suppose that G is a simplicial group, and that BG is equivalent to a simplicial set with finitely many non-degenerate simplices in each degree. Suppose that $(\mathcal{C}, \mathcal{C}_{>0})$ is a presentable ∞-category with a t-structure that is compatible with filtered colimits, right complete, and such that $\mathcal{C}_{<0}$ is compactly generated (e.g., $\mathcal{C}$ is coherent, or $\mathcal{C} = A\text{-mod}$ for a connective ring A). Then,*

$$(-)^G \colon \mathcal{C}^G \longrightarrow \mathcal{C}$$

preserves uniformly left t-bounded filtered colimits.

The group $G = \mathrm{SO}(2)$ has one more convenient property – since G is connected, any G-action on an object of a 1-category must be trivial. If a stable ∞-category $\mathcal{C}$ has a bounded t-structure, then objects of the heart generate – so if G acts on $\mathcal{C}$, then $\mathcal{C}^G$ is generated by objects with canonically trivial G-action. This observation is, for instance, a key step in showing that $\mathscr{R}(k\text{-mod}^{\mathrm{SO}(2)}) \simeq k[\![u]\!]\text{-mod}$ as mentioned in 1.3.2. We can encode these sort of arguments in the following Lemmas and Proposition:

Lemma 4.5.3. *Suppose that G is a connected simplicial group, and that BG is equivalent to a simplicial set with finitely many non-degenerate simplices in each degree. Suppose that $\mathcal{C}$ is a stable ∞-category with a t-structure and an action of G, and let $\theta \colon \mathcal{C}^G \to \mathcal{C}$ denote the natural functor. Then,*

- *(i) $\mathcal{C}^G$ carries a t-structure characterized by $(\mathcal{C}^G)_{>0} = \theta^{-1}(\mathcal{C}_{>0})$ and $(\mathcal{C}^G)_{<0} = \theta^{-1}(\mathcal{C}_{<0})$;*
- *(ii) The functor θ is exact, and induces an equivalence $(\mathcal{C}^G)^\heartsuit \simeq \mathcal{C}^\heartsuit$.*
- *(iii) If the t-structure on $\mathcal{C}$ is bounded (resp., compatible with filtered colimits, right complete, left complete, coherent), then so is the one on $\mathcal{C}^G$.*[9]

Lemma 4.5.4. *Let G be as above. Suppose that $\mathcal{C} \in \mathbf{dgcat}_k^{\mathrm{idm}}$ carries a bounded t-structure, and let G act on it* trivially. *Then, the natural functor*

$$\mathcal{C} \otimes_{\mathrm{Perf}\, k} \mathrm{Perf}\, C^*(BG, k) \longrightarrow \mathcal{C}^G$$

is an equivalence.

As noted above, regularity is not preserved by the formation of invariants. We can ask if there are G-invariants in the ∞-category $\mathbf{Reg}_t$. The answer is yes, as explained by the following Proposition:

Proposition 4.5.5. *Let G be as above.*

Suppose that $\mathcal{C} \in (\mathbf{Coh}_t)^G$ is an ∞-category with a G-action and a coherent t-structure. Then, $\mathscr{R}(\mathcal{C}^G) \to \mathscr{R}(\mathcal{C})$ identifes $\mathscr{R}(\mathcal{C}^G)$ as invariants for the G action on $\mathscr{R}(\mathcal{C})$ in the ∞-category $(\mathbf{Reg}_t)^G$. Furthermore, there is a natural equivalence

$$\mathrm{Ind}(\mathrm{Coh}(\mathcal{C})^G) \xrightarrow{\sim} \mathscr{R}(\mathcal{C}^G)$$

---

[8] By $k[B]$ we mean the free dg-commutative algebra, so that $B^2 = 0$ for grading reasons.↑

[9] Note that this is *not* true for regular.↑

Remark 4.5.6. Notice that combining Prop. 4.4.4 with one or both of Prop. 4.5.5 and Lemma 4.5.4 we can deduce a few useful statements.

For instance, let $\mathcal{C} = \mathrm{DCoh}(X_{\mathrm{dR}})$ denote the compact objects of $\mathrm{QC}^!(X_{\mathrm{dR}})$ and equip it with the trivial action of $G = \mathrm{SO}(2)$. By Prop. 4.4.4, $\mathcal{C}$ carries a bounded t-structure, so it follows from Lemma 4.5.4 that

$$\mathrm{DCoh}(X_{\mathrm{dR}})^{\mathrm{SO}(2)} \simeq \mathrm{DCoh}(X_{\mathrm{dR}}) \otimes_k k[\![u]\!]$$

and inverting u that

$$\mathrm{DCoh}(X_{\mathrm{dR}})^{\mathrm{Tate}} \simeq \mathrm{DCoh}(X_{\mathrm{dR}}) \otimes_k k(\!(u)\!)$$

Furthermore, combining with Prop. 4.5.5 allows us to conclude that

$$\mathscr{R}(\mathrm{QC}^!(X_{\mathrm{dR}})^{\mathrm{SO}(2)} \simeq \mathrm{Ind}(\mathrm{DCoh}(X_{\mathrm{dR}})^{\mathrm{SO}(2)} \simeq \mathrm{QC}^!(X_{\mathrm{dR}}) \otimes_k k[\![u]\!]$$

and inverting u that

$$\mathrm{QC}^!(X_{\mathrm{dR}})^{t\mathrm{Tate}} \simeq \mathrm{QC}^!(X_{\mathrm{dR}}) \otimes_k k(\!(u)\!)$$

Analogously, applying Prop. 4.5.5 to $\mathcal{C} = \mathrm{QC}^!(LX)$ – and then inverting u – we can conclude that

$$\mathrm{QC}^!(LX)^{t\mathrm{Tate}} \simeq \mathrm{Ind}(\mathrm{DCoh}(LX)^{\mathrm{Tate}}).$$

This shows that our "large category" Theorems are essentially equivalent to their "small category" variants.

4.6. ***t*-Tate-equivalences and functoriality.** For the remainder of this section $G = \mathrm{SO}(2)^r$, and $C^*(BG,k)_{loc} = C^*(BG,k)$ denotes the localization inverting all homogeneous elements. The goal of this subsection is to explain how to get actual *answers*, e.g., actual chain complexes, from the above formalism.

Lemma 4.6.1. *Suppose that $\mathcal{C}, \mathcal{D} \in \mathbf{Coh}_t$ are categories with coherent t-structures, and that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which is left t-exact up to a shift, preserves finite limits and colimits, and which preserves uniformly left t-bounded filtered colimits. Then,*

(i) There is an induced functor $\mathscr{R}(F)\colon \mathscr{R}(\mathcal{C}) \to \mathscr{R}(\mathcal{D})$ which is colimit preserving and left t-exact up to a shift, and fitting into the commutative diagram

$$\begin{tikzcd}
\mathscr{R}(\mathcal{C}) \arrow[r, "\mathscr{R}(F)"] \arrow[d, "\ell_d"'] & \mathscr{R}(\mathcal{D}) \arrow[d, "\ell_c"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

(ii) Let r_c (resp., r_d) be the right adjoint to ℓ_c (resp., ℓ_d). Then, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{<\infty} \arrow[r, "F"'] \arrow[d, "r_d"'] & \mathcal{D}_{<\infty} \arrow[d, "r_c"] \\
\mathscr{R}(\mathcal{C}) \arrow[r, "\mathscr{R}(F)"'] & \mathscr{R}(\mathcal{D})
\end{tikzcd}$$

Definition 4.6.2. Suppose that $\mathcal{C} \in \mathbf{Coh}_t^G$. We say that a morphism $f \in \mathcal{C}^G$ is a t-Tate-equivalence if it becomes an equivalence after applying the composite functor

$$\mathcal{C}^G \xrightarrow{r_c} \mathscr{R}(\mathcal{C}^G) \longrightarrow \mathcal{C}^{t\mathrm{Tate}}.$$

Remark 4.6.3. It is easy to see from the definitions, and Lemma 4.6.1, that if F is a G-equivariant functor satisfying the hypotheses of Lemma 4.6.1 then $F^{t\mathrm{Tate}}$ makes sense and preserves t-Tate-equivalences between left t-bounded objects. Furthermore, both the formation of $F \mapsto \mathscr{R}(F)$ and $F \mapsto F^{t\mathrm{Tate}}$ are compatible with composition.

Suppose now that $F\colon \mathcal{C} \to \mathcal{D}$ is G-equivariant functor between categories with G-action, and that we'd like to understand some Tate approximation to F. Assume that $\mathcal{C}, \mathcal{D}$ are both regular and that the G action on $\mathcal{D}$ is *trivial*. Then, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^G_{<\infty} \arrow[r] \arrow[d, "F"'] & \mathscr{R}(\mathcal{C}^G) \arrow[r] \arrow[d, "\mathscr{R}(F)"'] & \mathcal{C}^{t\mathrm{Tate}} \arrow[d, "F^{t\mathrm{Tate}}"'] \\
\mathcal{D}^G_{<\infty} \arrow[r] \arrow[dr, "(-)^G"'] & \mathscr{R}(\mathcal{D}^G) \arrow[r] \arrow[d, equal] & \mathcal{D}^{t\mathrm{Tate}} \arrow[d, equal] \\
 & C^*(BG)\text{-mod}(\mathcal{D}) \arrow[r] & C^*(BG)_{loc}\text{-mod}(\mathcal{D})
\end{tikzcd}$$

So on homologically bounded above objects, $\mathscr{R}(F)$ is nothing more than F combined with ordinary invariants functor with its $C^*(BG)$-action; and $F^{t\mathrm{Tate}}$ analogously with the ordinary Tate homology.

5. LOCALIZATION THEOREMS

5.1. **Formulation of the "Localization Theorem".**

Notation 5.1.1.

- To a (finite) space S, we can associate the (derived) algebraic space

$$X^S \overset{\text{def}}{=} \mathrm{Map}(S, X) \qquad X^S(\operatorname{Spec} R) = \mathrm{Map}_{\mathbf{sSet}}(S, X(\operatorname{Spec} R)).$$

  This functor is representable by an algebraic space, when X is an algebraic space.
- The projection $S \to \mathrm{pt}$ endows X^S with a natural map from X, $X^{\mathrm{pt}} \to X^S$. Define $\widehat{X^S}$ to be the completion of X^S along X:

$$\widehat{X^S}(\operatorname{Spec} R) \overset{\text{def}}{=} X^S \times_{(X^S)_{\mathrm{dR}}} X_{\mathrm{dR}} = \text{Commutative diagrams of the form}\quad \begin{tikzcd}
\operatorname{Spec} R_{\mathrm{red}} \arrow[r, dashed] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r, dashed] & X^S
\end{tikzcd}$$

- More generally, the constructions $S \mapsto X^S$ and $S \mapsto \widehat{X^S}$ are contravariantly functorial in S.

5.1.2. The construction $S \mapsto \widehat{X^S}$ behaves qualitatively differently in the following three cases:

- If S is non-empty, then $X \to X^S$ is a closed-immersion. In this case, $\widehat{X^S}$ is a subfunctor of X^S: $\widehat{X^S}(\operatorname{Spec} R)$ just consists of those components of $X^S(\operatorname{Spec} R)$ which set-theoretically (as opposed to scheme-theoretically) factor through the diagonal.
- If $S = \emptyset$, then $X^S = (X^S)_{\mathrm{dR}} = \mathrm{pt}$. So, $\widehat{X^S} = X_{\mathrm{dR}}$.
- If S is non-empty and *connected*, and $X(R_{\mathrm{red}})$ is discrete (e.g., X a derived algebraic space), then $X \to X^S$ is set-theoretically a bijection. In this case, $\widehat{X^S} \to X^S$ is an equivalence.

Our main Localization Theorem will be:

Theorem 5.1.3 ("Localization Theorem for $\mathrm{QC}^!(\widehat{X^-})$")**.** *Suppose that $G = \mathrm{SO}(2)^r$ is a compact torus, that S is a finite G-space, and that $F \hookrightarrow S$ is the inclusion of the fixed locus. The inclusion of fixed points gives rise to a map $p\colon X^S \to X^F$ of derived schemes under X, and consequently to a map*

$$\widehat{p}\colon \widehat{X^S} \to \widehat{X^F}$$

of their completions along X. The functors $\widehat{p}_*$ and $\widehat{p}^!$ are G-equivariant, and left t-exact up to a shift, and give rise to adjoint equivalences

$$(\widehat{p}_*)^{t\mathrm{Tate}}\colon\ \mathrm{QC}^!(\widehat{X^S})^{t\mathrm{Tate}} \leftrightarrow \mathrm{QC}^!(\widehat{X^F})^{t\mathrm{Tate}}\colon (\widehat{p}^!)^{t\mathrm{Tate}}$$

5.2. **Reducing "loop spaces and connections" to the "Localization Theorem".** Let us explain how to view Theorem 1.3.5 as a special case of Theorem 5.1.3:

Example 5.2.1. We can reformulate the map π of Theorem 1.3.5: Consider the (SO(2)-equivariant) inclusion $\emptyset \hookrightarrow S^1$. It gives rise to a projection, under X, $\pi\colon \widehat{X^{S^1}} \to \widehat{X^{\emptyset}}$. From 5.1.2, we may identify $\widehat{X^{S^1}} \simeq \widehat{X^{S^1}} = \widehat{LX}$ and $\widehat{X^{\emptyset}} \simeq X_{\mathrm{dR}}$. Under these identifications, this is equivalent to the map π described earlier.

Assuming Theorem 5.1.3, which we will give in the next subsection, we can of course complete the proof of Theorem 1.3.5:

Proof of Theorem 1.3.5. Apply Theorem 1.3.5 to the inclusion $\emptyset \hookrightarrow S^1$ of $G = \mathrm{SO}(2)$ fixed points.

To conclude it is enough to prove that

$$\mathrm{QC}^!(X_{\mathrm{dR}})^{f\mathrm{Tate}} \simeq \mathrm{QC}^!(X_{\mathrm{dR}})(\!(\beta)\!)$$

which follows by Remark 4.5.6. ☐

Remark 5.2.2. We have thus recast our form of "loop spaces and connections" Theorem 1.3.5 as a special case of the "*Localization* Theorem" Theorem 5.1.3, applied to the inclusion of SO(2) fixed points $\emptyset \hookrightarrow S^1$. Although Theorem 5.1.3 seems to strictly generalize the original Theorem 1.3.5 we are not aware of any other examples that are as interesting.

5.3. **Reduction to Localization for Dualizing Complexes.** In order to prove Theorem 5.1.3, we will reduce it to a Localization Theorem one category-level down. To do this, we need the following more algebraic description of $\mathrm{QC}^!(\widehat{X^S})$ making use of the good formal properties of $\mathrm{QC}^!$:

Proposition 5.3.1. *Suppose S is a (finite) space, and let ΣS denote the un-reduced suspension of S. Then,*

(i) *The inclusion of cone points $S^0 = \Sigma\emptyset \hookrightarrow \Sigma S$ gives rise to a projection $p'\colon \widehat{X^{\Sigma S}} \to \widehat{X^{S^0}} \to X^2$. Then,*

$$(p')_* \omega_{\widehat{X^{\Sigma S}}} \in \mathrm{QC}^!(X^2)$$

naturally lifts to an algebra object with respect to the shriek-convolution product $\overset{!}{\circ}$ on $\mathrm{QC}^!(X^2)$. By abuse of notation, call this algebra object $\omega_{\widehat{X^{\Sigma S}}}$.[10]

(ii) *The natural map $c\colon X \to \widehat{X^S}$ induces an equivalence*

$$c^!\colon\ \mathrm{QC}^!(\widehat{X^S}) \xrightarrow{\sim} (c^! c_*)\text{-mod}\,\mathrm{QC}^!(X)$$

Furthermore, the monad $(c^! c_*) \in \mathbf{Alg}(\mathrm{Fun}^L(\mathrm{QC}^!(X), \mathrm{QC}^!(X)))$ identifies with $\omega_{\widehat{X^{\Sigma S}}} \in \mathbf{Alg}(\mathrm{QC}^!(X^2))$ under the the !-Fourier-Mukai equivalence Theorem 3.2.1.

Proof.

(i) Note that the ∞-category $\mathbf{sSet}_{S^0/}$ of spaces under $S^0 = \{+1, -1\}$ is equipped with a monoidal structure via wedge product:

$$(X, x_-, x_+) \wedge (Y, y_-, y_+) = ((X \sqcup Y)/(x_+ \sim y_-), x_-, y_+)$$

There is a functor $S \mapsto \Sigma S$ from un-pointed spaces to co-A_∞ monoids in this category: the distinguished points are the two cone points of the suspension, and the co-monoid structure is by "pinching map" $\Sigma S \to (\Sigma S) \wedge (\Sigma S)$. One can check that this structure gives rise to the requisite algebra under convolution product.

(ii) (C.f. [P1, Section 5]) The Cech nerve of $X \to \widehat{X^S}$ identifies with the result of applying $\widehat{X^{-}}$ to the co-augmented co-simplicial space associated to the co-A_∞ monoid ΣS above. To see this, note that $S \mapsto \widehat{X^S}$ takes finite colimits (under the initial object $\emptyset$) to finite limits (over $\widehat{X^{\emptyset}} = X_{\mathrm{dR}}$). For instance, at the level of 1-simplices this is the fact that $X \times_{\widehat{X^S}} X = \widehat{X^{\Sigma S}}$.

We claim that $c^!$ admits a left adjoint c_* compatible with base-change, and that it is monadic. This should be regarded as a mild generalization of Theorem 3.3.2 where the map c is "ind-finite."

---

[10] We do not believe that this abuse of notation is harmful: This is the only place where it is naturally an algebra object! The only potential ambiguity is in $\mathrm{QC}^!(\widehat{X^2}) = \mathrm{QC}^!_X(X^2)$ versus $\mathrm{QC}^!(X^2)$, but that is a full subcategory.↑

Indeed, the result can be found worded in this style in [GR4] – for the this generality is needed to generalize the results of this note to X a stack, but for X an algebraic space we can get by with less:

In the case where S is non-empty and X is separated, we have that c is the completion of an ordinary (locally) closed immersion. Thus, the existence of $c^!$ follows from the usual functoriality for $\mathrm{QC}^!$ in the (easier) quasi-affine case. Furthermore, the monadicity follows from the descent/completion paradigm in Theorem 3.3.2.

Meanwhile in the case of $S = \emptyset$ we are studying the map $X \to X_{\mathrm{dR}}$ – and this is now the assertion that crystals are monadic over their "right realization" i.e., the forgetful functor from crystals to $\mathrm{QC}^!(X)$. This is étale local on X, so that we may suppose that X is affine and embedded as a closed subscheme of a smooth affine variety M. It is clear that $\widehat{M}_{\mathrm{dR}} = X_{\mathrm{dR}}$. Consider now the map $f\colon \widehat{M} \to \widehat{M}_{\mathrm{dR}}$, both completions taken along X. We claim that $\widehat{M}_{\mathrm{dR}}$ is actually the geometric realization, in étale sheaves, of the Cech nerve of f – the realization identifies with the subfunctor of those points which étale locally admit a factorization through f, so it is enough to show that any $\eta_{\mathrm{red}} \in (\widehat{M}_{\mathrm{dR}})(R)$ extends to a diagram

$$\begin{tikzcd}
(\operatorname{Spec} R)_{\mathrm{red}} \arrow[r, "\eta_{\mathrm{red}}"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r, dashed, "\eta"'] & M
\end{tikzcd}$$

corresponding to an R-point of $\widehat{M}$. But this follows by formal smoothness of M.

Using this and the completion-support paradigm of Theorem 3.3.2 we can show $\mathrm{QC}^!(X_{\mathrm{dR}}) \simeq \mathrm{QC}^!_X(M_{\mathrm{dR}})$ ("Kashiwara's Lemma"), and thus construct c_* and check the Barr-Beck conditions for monadicity.

Continuing, the Beck-Chevalley condition [L2, Section 6.2.4] ensures that the monad is given by the formal groupoid over X associated to $\widehat{X^{\Sigma S}}$:

$$\begin{tikzcd}
X \arrow[r, shift left=1.5] & \widehat{X^{\Sigma S}} \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left=3] \arrow[r, shift right=1.5] & \cdots \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3]
\end{tikzcd}$$

To conclude, apply the projection formula for $\mathrm{QC}^!$:

$$(p_2 \circ p)_*(p_2 \circ p)^! = (p_2)_*(p')_* \left((p')^!(p_1)^!(-) \overset{!}{\otimes} \omega_{\widehat{X^{\Sigma S}}} \right) = (p_2)_* \left((p_1)^!(-) \overset{!}{\otimes} (p')_* \omega_{\widehat{X^{\Sigma S}}} \right) \qquad \square$$

Remark 5.3.2. Explicit simplicial models for the structure in (i) above can be obtained by considering the edgewise subdivision of the simplicial circle S^1. In this form, this construction has figured prominently in the geometric formulations of the decomposition results for Hochschild homology of commutative algebras via power operations.

Example 5.3.3. Let's spell out the above in our two cases of interest: $S = \emptyset$ and $S = S^1$.

- Suppose $S = \emptyset$, so that we are interested in $\mathrm{QC}^!(\widehat{X^S}) = \mathrm{QC}^!(X_{\mathrm{dR}})$, i.e., right D-modules on X. We have $\Sigma S = S^0$, so that n-simplices of the resulting simplicial diagram are $\widehat{X^{S^0}} = X^n{}^{\wedge}_{X}$, etc. This is the usual simplicial presentation of the de Rham stack, as the quotient of X by the infinitesimal groupoid. Passing to $\mathrm{QC}^!(-)$, we obtain
$$\mathrm{QC}^!(X_{\mathrm{dR}}) = \mathrm{Tot}\left\{ \mathrm{QC}^!(\widehat{X^{\bullet+1}}) \right\} = \mathrm{Tot}\left\{ \mathrm{QC}^!_X(X^{\bullet+1}) \right\} = \omega_{\widehat{X^{S^0}}}\text{-mod}\, \mathrm{QC}^!(X)$$
Note that $\widehat{X^{S^0}}$ is just the formal completion of X^2 along the diagonal, and $\omega_{\widehat{X^{S^0}}}$ may be realized as the directed limit of ω_{X_n} where X_n runs through a suitable family of nilthickenings. In case X is smooth, we can thus directly identify $\omega_{\widehat{X^{S^0}}}$ and $\omega_X \otimes_{\mathcal{O}_X} D_X$ as algebras in $\mathrm{Fun}^L(\mathrm{QC}^!(X), \mathrm{QC}^!(X))$.
- Suppose $S = S^1$, so that we are interested in $\mathrm{QC}^!(\widehat{X^S}) = \mathrm{QC}^!(\widehat{LX})$. The 1-simplices of the resulting simplicial diagram are $\widehat{X^{S^2}}$, and the n-simplices are $X^{\widehat{S^2 \vee \cdots \vee S^2}}$ where the superscript is taken n-times.

Passing to $\mathrm{QC}^!(-)$, we obtain

$$\mathrm{QC}^!(\widehat{LX}) = \mathrm{Tot}\left\{\mathrm{QC}^!\left(X^{\widehat{S^2\wedge\cdots\wedge S^2}}\right)\right\} = \omega_{\widehat{X^{S^2}}}\text{-mod}\,\mathrm{QC}^!(X)$$

- Let

$$\phi\colon \omega_{\widehat{X^{S^2}}} \longrightarrow \omega_{\widehat{X^{S^0}}} = \underline{\mathrm{R}\Gamma}_X(\omega_{X^2})$$

denote the natural SO(2)-equivariant map in $\mathbf{Alg}(\mathrm{QC}^!(X^2), \overset{!}{\circ})$. Under the above identifications, the SO(2)-equivariant adjoint pair $(\pi_*, \pi^!)$ of the Theorem may be identified with induction and restriction along ϕ:

$$\begin{tikzcd}
\mathrm{QC}^!(\widehat{LX}) \arrow[r, shift left, "i_*"] \arrow[d, "q^!"', "\sim"] & \mathrm{QC}^!(X_{\mathrm{dR}}) \arrow[l, shift left, "i^!"] \arrow[d, "\sim"', "c^!"] \\
\omega_{\widehat{X^{S^2}}}\text{-mod}(\mathrm{QC}^!(X)) \arrow[r, shift left, "\mathrm{Ind}_\phi"] & \omega_{\widehat{X^{S^0}}}\text{-mod}(\mathrm{QC}^!(X)) \arrow[l, shift left, "\mathrm{Res}_\phi"]
\end{tikzcd}$$

Using these, one can deduce Theorem 5.1.3 from the following decategorified Localization Theorem, which we will prove in the next subsection:

Theorem 5.3.4 ("Localization Theorem for $\omega_{\widehat{X^-}}$"). *Suppose $G = \mathrm{SO}(2)^r$ is a compact torus, that S is a finite G-space, and that $F \hookrightarrow S$ is the inclusion of the fixed locus. This gives rise to a map $\widehat{p}\colon \widehat{X^S} \to \widehat{X^F}$ of derived schemes under X and a natural map*

$$\widehat{\alpha}_{F,S}\colon p_*\omega_{\widehat{X^S}} \longrightarrow \omega_{\widehat{X^F}} \in \mathrm{QC}^!(\widehat{X^F})$$

If the pointed G-space S/F admits the structure of connected co-H-space in pointed G-spaces, then the map $\widehat{\alpha}_{F,S}$ induces an equivalence upon passing to Tate constructions.

Assuming Theorem 5.3.4, let us complete the proof of Theorem 5.1.3:

Proof of Theorem 5.1.3. We first show that

$$\phi^{\mathrm{Tate}}\colon (\omega_{\widehat{X^{\Sigma S}}})^{\mathrm{Tate}} \to (\omega_{\widehat{X^{\Sigma F}}})^{\mathrm{Tate}}$$

is an equivalence in $\mathbf{Alg}(\mathrm{QC}^!(X^2))$. Since the forgetful functor $\mathbf{Alg}(\mathrm{QC}^!(X^2)) \to \mathrm{QC}^!(X^2)$ is conservative and preserves limits (such as taking S^1-invariants) and filtered colimits (such as passing from invariants to Tate construction), it is enough to check that the analogous map in $\mathrm{QC}^!(X^2)$ is an equivalence. This map is in fact pushed forward from $\mathrm{QC}^!(\widehat{X^{\Sigma F}})$, where it is an equivalence by applying Theorem 5.3.4 to $\Sigma F \hookrightarrow \Sigma S$: Note that $(\Sigma S)/(\Sigma F) \simeq S^1 \wedge (S/F)$ as pointed G-spaces; consequently, it admits a co-H-space structure induced by the pinch map on S^1.

In light of Prop. 4.5.5, it is now enough to show that

$$(p_*)^{\mathrm{Tate}}\colon \mathrm{DCoh}(\widehat{X^S})^{\mathrm{Tate}} \longrightarrow \mathrm{DCoh}(\widehat{X^F})^{\mathrm{Tate}}$$

is an equivalence. Note that $\mathrm{DCoh}(\widehat{X^S})^{\mathrm{Tate}}$ (and similarly for X^F) is generated under cones, shifts, and retracts by the images of objects of $\mathrm{DCoh}(X)^\heartsuit$ equipped with the trivial G-action. It thus suffices to show that $(p_*)^{\mathrm{Tate}}$ is fully faithful on pushforwards along $q\colon X \to \widehat{X^S}$. That is, we must show that

$$\mathrm{RHom}_{\mathrm{QC}^!(\widehat{X^S})}(q_*\mathscr{F}, q_*\mathscr{G})^{\mathrm{Tate}} \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(\widehat{X^F})}(p_*q_*\mathscr{F}, p_*q_*\mathscr{G})^{\mathrm{Tate}}$$

is an equivalence for all $\mathscr{F}, \mathscr{G} \in \mathrm{DCoh}\,X$. But we may identify this map with each of

$$\mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{F}, \Phi^!_{\omega_{\widehat{X^{\Sigma S}}}}\mathscr{G})^{\mathrm{Tate}} \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{F}, \Phi^!_{\omega_{\widehat{X^{\Sigma F}}}}\mathscr{G})^{\mathrm{Tate}}$$

$$\mathrm{RHom}_{\mathrm{QC}^!(X^2)}(\mathscr{F} \boxtimes \mathbb{D}\mathscr{G}, \omega_{\widehat{X^{\Sigma S}}})^{\mathrm{Tate}} \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(X^2)}(\mathscr{F} \boxtimes \mathbb{D}\mathscr{G}, \omega_{\widehat{X^{\Sigma F}}})^{\mathrm{Tate}}$$

$$\mathrm{RHom}_{\mathrm{QC}^!(X^2)}(\mathscr{F}\boxtimes\mathbb{D}\mathscr{G}, (\omega_{\widehat{X^{\Sigma S}}})^G)\otimes_{C^*(BG)}C^*(BG)_{loc} \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(X^2)}(\mathscr{F}\boxtimes\mathbb{D}\mathscr{G}, (\omega_{\widehat{X^{\Sigma F}}})^G)\otimes_{C^*(BG)}C^*(BG)_{loc}$$

Since $\mathscr{F} \boxtimes \mathbb{D}\mathscr{G}$ is compact in $\mathrm{QC}^!(X^2)$, we can move the colimit inside – this completes the proof. ☐

5.4. **Proof of Localization for Dualizing Complexes.** Let us record the following equivariant version of Prop. 2.3.2. Note that (i) and (ii) are immediate from the functoriality considerations there, while (iii) follows from the generality of the proof:

Proposition 5.4.1. *Suppose that X is a derived algebraic space in characteristic zero, and that G is a connected simplicial group which is equivalent to a finite space (e.g., $G = \{\mathrm{id}\}$ or $G = \mathrm{SO}(2)$). Then,*

- *(i) A finite, pointed, G-space $pt \to S$ gives rise to a map $p\colon X^S \to X$ in derived schemes under X with G-action. This gives rise to a G-equivariant $\mathcal{O}_X$-module $p_*\mathcal{O}_{X^S}$ functorially in pointed G-maps of S. More precisely, the above determines a composite sequence of functors*

$$\mathbf{Sp}_*^G \xrightarrow{X^S} \mathrm{DerSch}^G_{X//X} \xrightarrow{p_*} \mathrm{QC}(X)^G$$

- *(ii) There is a decreasing filtration on $p_*\mathcal{O}_{X^S} \in \mathrm{QC}(X)^G$, functorial in pointed G-maps of S,*

$$p_*\mathcal{O}_{X^S} = F^0 \supset F^1 \supset F^2 \supset \cdots$$

whose associated graded pieces can be naturally identified with

$$\mathrm{gr}_i F^\bullet \simeq \mathrm{Sym}^i\left(\mathbb{L}_{X/X^S}\right) \simeq \mathrm{Sym}^i\left(\mathbb{L}_X \otimes_k \widetilde{C}_\bullet(S)\right)$$

where $\widetilde{C}_\bullet(S)$ denotes reduced k-linear chains on S.

- *(iii) Suppose that S admits a co-H-space structure as pointed G-space (e.g., S is the suspension of a pointed G-space). Then, the filtration of (ii) splits and there is an equivalence of G-equivariant $\mathcal{O}_X$-modules*

$$p_*\mathcal{O}_{X^S} = \mathcal{O}_X \otimes R \xrightarrow{\sim} \mathrm{Sym}_{\mathcal{O}_X}\left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right)$$

Up to duality, the point is now that $\omega_{\widehat{X^S}}$ and $\omega_{\widehat{X^F}}$ carry filtrations with the property that on each associated graded piece we know that we have a Tate equivalence by Atiyah-Bott Localization. If Tate equivalences were stable under arbitrary inverse limits, we would be done! Unfortunately, this is not true. What *is* true is that if at each level of an inverse limit the cone on SO(2)-invariants is killed by u^N, for a *fixed* element u and integer N, then we might expect the limit to still be a Tate equivalence. That's why we'll end up needing to use the last, and most subtle, part of the previous Proposition. We formalize this observation about "being killed by u^N" being preserved by many things:

Lemma 5.4.2. *Let $G = \mathrm{SO}(2)^r$ be a compact torus, and fix a non-zero homogeneous element $u \in H^*(BG, k)$. Consider the full-subcategory $\mathbf{Tors}_N \subset (k\text{-mod})^G$*

$$(k\text{-mod})^G \supset \mathbf{Tors}_N = \{V \colon \textit{The object } V^G \in C^*(BG,k)\text{-mod} \textit{ is } u^N\textit{-torsion}\}$$

Then,

- *(i) $\mathbf{Tors}_N$ is stable under arbitrary products;*
- *(ii) $\mathbf{Tors}_N$ is stable under retracts;*
- *(iii) $\mathbf{Tors}_N$ is stable under uniformly bounded above sums;*
- *(iv) If $V, V' \in (k\text{-mod})^G$ are both bounded above and $V \in \mathbf{Tors}_N$, then $V \otimes V' \in \mathbf{Tors}_N$.*
- *(v) If $V \in \mathbf{Tors}_N$ is dualizable (i.e., k-perfect), then $V^\vee \in \mathbf{Tors}_N$.*

Proof.

- (i) Suppose h_α is a null-homotopy of u^N on $(V_\alpha)^G$ for each α in some indexing set. Then, $\prod h_\alpha$ is a null-homotopy of u^N on

$$\prod_\alpha (V_\alpha)^G \simeq \left(\prod_\alpha V_\alpha\right)^G$$

- (ii) Obvious.
- (iii) Suppose $V_\alpha \in (k\text{-mod})^G$ is a uniformly bounded above family, and that h_α is a null-homotopy of u^N on $(V_\alpha)^G$ for each α. Then, the natural map

$$\bigoplus_\alpha (V_\alpha)^G \longrightarrow \left(\bigoplus_\alpha V_\alpha\right)^G$$

is an equivalence (c.f., Lemma 4.5.2) and $\oplus h_\alpha$ gives the desired null-homotopy of u^N on the sum.

(iv) It suffices to note that the natural map

$$V^G \otimes_{k^G} (V')^G \to (V \otimes_k V')^G$$

is an equivalence under the boundedness assumptions (c.f., the discussion after Lemma 4.5.2). For then, suppose that h is a null-homotopy of u^N on V^G; then $h \otimes 1$ is a null-homotopy of $u^N = u^N \otimes 1$ on the tensor product.

(v) Since V and $V^\vee$ are both k-perfect, they are bounded above. The duality datum between V and $V^\vee$ then induces a $k[u]$-linear duality datum between V^G and $(V^\vee)^G$. The result then follows, since the dual morphism to u^N on V^G is u^N on $(V^\vee)^G$; a null-homotopy of the former gives rise to a null-homotopy of the latter. ☐

Putting these facts together, we are ready to explain our decategorified Localization Theorem:

Proof of Theorem 5.3.4.

Reduction to $F = \mathbf{pt}$: Consider the "big diagonal" $i \colon X \to \widehat{X^F}$ induced by the projection $F \to \mathrm{pt}$. Since $i^! \colon \mathrm{QC}^!(\widehat{X^F}) \to \mathrm{QC}^!(X)$ is conservative and commutes with limits and colimits (c.f., the proof of Prop. 5.3.1) it is enough to show that

$$i^! \widehat{\alpha}_{F,S} \colon i^! p_* \omega_{\widehat{X^S}} \longrightarrow i^! \omega_{\widehat{X^F}} \in \mathrm{QC}^!(X)$$

induces an equivalence upon passing to Tate constructions. Considering the Cartesian square

$$\begin{tikzcd}
\widehat{X^{S/F}} \arrow[r] \arrow[d, "p'"] & \widehat{X^S} \arrow[d, "\widehat{p}"] \\
X \arrow[r, "i"'] & \widehat{X^F}
\end{tikzcd}$$

we see that base-change[11] identifies $i^! \widehat{\alpha}_{F,S}$ with

$$\widehat{\alpha}_{\mathrm{pt},S/F} \colon (p')_* \omega_{\widehat{X^{S/F}}} \longrightarrow \omega_X$$

Thus we are reduced to proving the result for the inclusion $F = \mathrm{pt} \hookrightarrow S$, with S a connected co-H-space in pointed G-spaces on which G acts without fixed points away from the base-point.

Converting to algebra: In particular, note that we are now in a situation where Prop. 2.3.2(iii) applies.

We must show that for all $\mathscr{K} \in \mathrm{DCoh}(X)$ the natural map

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{K}, \omega_X) \longleftarrow \mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{K}, \omega_{X^S}) &= \mathrm{RHom}_{\mathrm{QC}^!(X)}(\mathscr{K}, F_R \, \mathrm{RHom}^{\otimes}_{\mathrm{QC}(X)}(p_* \mathcal{O}_{X^S}, \omega_X)) \\
&= \mathrm{RHom}_{\mathrm{QC}(X)}(\mathscr{K} \otimes_{\mathcal{O}_X} p_* \mathcal{O}_{X^S}, \omega_X)
\end{aligned}$$

becomes an equivalence after applying the Tate construction. Applying Prop. 2.3.2(iii) to S, we see that this is equivalent to showing that

$$\mathrm{RHom}_{\mathrm{QC}(X)} \left(\mathscr{K} \otimes \mathrm{Sym}^{\bullet \geq 1}_{\mathcal{O}_X} \left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S) \right), \omega_X \right) \in \mathrm{Cpx}_k^G$$

becomes contractible after passing to Tate construction. (Recall that $\widetilde{C}_\bullet(S)$ denotes reduced chains.)

Completing the proof: The property for a complex $V \in (k\text{-mod})^G$ to have contractible Tate construction is not generally stable under infinitary operations, like those seemingly involved in forming the above complex from $\widetilde{C}_\bullet(S)$. Fortunately in our case, we are in the stronger situation of Lemma 5.4.2 above: By the (proof of the) Atiyah-Bott Localization Theorem, there exists an element $u \in H^*(BG, k)$ and an $N > 0$ such that $\widetilde{C}_\bullet(S) \in \mathbf{Tors}_N$.

Working backwards towards our goal using the various assertions of Lemma 5.4.2:

---

[11] The reader who is concerned about base change with so many formal spaces in sight should be comforted: We only care about the case where F, and thus S and S/F, is non-empty so that the pushforward along the inclusion determines a fully faithful functor $\mathrm{QC}^!(\widehat{X^F}) \hookrightarrow \mathrm{QC}^!(X^F)$. So, we are free to work with "non-completed spaces" but "completed sheaves" (more precisely, sheaves set-theoretically supported along the big diagonal).↑

- By (v), or the usual cohomological form of Atiyah-Bott, $\widetilde{C}^\bullet(S) \in \mathbf{Tors}_N$.
- By (iv), $\widetilde{C}^\bullet(S)^{\otimes i} \in \mathbf{Tors}_N$ for $i \geq 1$.
- By (iv) again (or (iii)),

$$\begin{aligned}\mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right)^{\otimes_{\mathcal{O}_X} i}, \omega_X\right) &\simeq \mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathbb{L}_X^{\otimes_{\mathcal{O}_X} i} \otimes_k \widetilde{C}_\bullet(S)^{\otimes i}, \omega_X\right) \\ &\simeq \mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathbb{L}_X^{\otimes_{\mathcal{O}_X} i} \otimes_k \widetilde{C}_\bullet(S)^{\otimes i}, \omega_X\right) \\ &\simeq \mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathbb{L}_X^{\otimes_{\mathcal{O}_X} i}, \omega_X\right) \otimes_k \widetilde{C}^\bullet(S)^{\otimes i} \in \mathbf{Tors}_N\end{aligned}$$

  for $i \geq 1$. Here we have implicitly used that $\widetilde{C}_\bullet(S)$ is perfect over k to dualize it out. Furthermore, in concluding we used that $\mathscr{K} \otimes_{\mathcal{O}_X} \mathbb{L}_X^{\otimes i}$ is almost perfect and ω_X is bounded above to conclude that the RHom was bounded above.
- By (ii),

$$\mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathrm{Sym}^i\left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right), \omega_X\right) \in \mathbf{Tors}_N$$

  for $i \geq 1$, as it is a retract of the previous step (recall that we are in characteristic zero).
- By (i),

$$\begin{aligned}\mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{O_X} \mathrm{cone}\left\{\mathcal{O}_X \to \mathcal{O}_{X^S}\right\}, \omega_X\right) &\simeq \mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathrm{Sym}^{\bullet \geq 1}\left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right), \omega_X\right) \\ &= \prod_{i \geq 1} \mathrm{RHom}_{\mathrm{QC}(X)}\left(\mathscr{K} \otimes_{\mathcal{O}_X} \mathrm{Sym}^i\left(\mathbb{L}_X \otimes \widetilde{C}_\bullet(S)\right), \omega_X\right) \in \mathbf{Tors}_N\end{aligned}$$

We have thus proven that our given complex is in $\mathbf{Tors}_N$, from which it follows that it becomes contractible after passing to Tate constructions. $\square$

6. Applications: The Borel-Moore version of the Theorem of Feigin-Tsygan

6.1. D-modules and Borel-Moore chains.

The following is presumably well-known, but we do not know of a reference:

Theorem 6.1.1. *Suppose that X is a derived algebraic space. Then, there exists a universal trace morphism*[12]

$$\mathrm{tr}\colon \mathbf{HH}_\bullet(\mathrm{DCoh}\, X_{\mathrm{dR}})_{\mathrm{SO}(2)} \to C_\bullet^{dR,BM}(X)$$

inducing an $\mathrm{SO}(2)$*-equivariant equivalence*

$$\mathbf{HH}_\bullet(\mathrm{DCoh}\, X_{\mathrm{dR}}) \simeq C_\bullet^{dR,BM}(X)$$

where the right-hand side has the trivial $\mathrm{SO}(2)$*-action.*

Proof. The key fact is that shriek integral transforms give an equivalence [G2]

$$\Phi^!\colon\ \mathrm{QC}^!((X^2)_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{Fun}^L(\mathrm{QC}^!(X_{\mathrm{dR}}), \mathrm{QC}^!(X_{\mathrm{dR}}))$$

under which the identify correspnods to $\Phi^!_{\Delta_* \omega_X}$ and the trace corresponds to

$$\mathrm{tr}(\Phi^!_{\mathscr{K}}) = \mathrm{R\Gamma}(X_{\mathrm{dR}}, \Delta^! \mathscr{K})$$

where $\mathrm{R\Gamma}(X_{\mathrm{dR}}, -)$ denotes the D-module pushforward to a point (*not* flat sections). In particular,

$$\mathrm{tr}(\mathrm{id}_{\mathrm{QC}^!(X_{\mathrm{dR}})}) = \mathrm{R\Gamma}(X_{\mathrm{dR}}, \Delta^! \Delta_* \omega_{X_{\mathrm{dR}}}) \simeq \mathrm{R\Gamma}(L(X_{\mathrm{dR}}), \omega_{L(X_{\mathrm{dR}})})$$

But $L(X_{\mathrm{dR}}) = X_{\mathrm{dR}}$. Done more systematically, as in Theorem 6.3.2 below, this gives an SO(2)-equivariant equivalence $\mathbf{HH}_\bullet(\mathrm{DCoh}\, X_{\mathrm{dR}}) \simeq C_\bullet^{dR,BM}(X)$. $\square$

---

[12] The left-hand side is the coinvariants for the SO(2)-action on Hochschild chains, i.e,. it is the cyclic cochains↑

Remark 6.1.2. If X is smooth, then $\operatorname{DCoh} X_{\mathrm{dR}}$ contains a full-subcategory $\operatorname{Loc} X$ of (de Rham) local systems on X: i.e., those D-modules whose underlying $\mathcal{O}_X$-module is perfect. Using the Riemann-Hilbert correspondence to identify $\operatorname{Loc} X$ with topological local systems, one can construct an $\mathrm{SO}(2)$-equivariant identification $\mathbf{HH}_\bullet(\operatorname{Loc} X) \simeq C^\bullet_{sing}(LX)$ – in particular, the $\mathrm{SO}(2)$-action *is* non-trivial.

One might ask for a suitable form of the Atiyah-Bott Localization Theorem to hold for LX, and its fixed locus X, allowing us to compate $\mathbf{HP}_\bullet(\operatorname{Loc} X)$ to $C^\bullet_{sing}(X) \otimes_k k(\!(u)\!)$. If X is not simply connected, this is quite unreasonable – e.g., if $X = \mathbb{G}_m$ then $\operatorname{Loc} X = (\operatorname{Perf} k)^{\mathbb{Z}}$.

6.2. **Detour – decategorified consequence.** It's time to verify that we can actualy get some handle on the functoriality promised in Theorem 5.1.3.

Proposition 6.2.1. *Suppose that* $\mathscr{F} \in \mathrm{QC}^!(X_{\mathrm{dR}})_{<\infty}$. *Then, the natural map*

$$\mathrm{R\Gamma}(LX, \pi^! \mathscr{F})^{\mathrm{Tate}} \longrightarrow \mathrm{R\Gamma}(X_{\mathrm{dR}}, \mathscr{F})^{\mathrm{Tate}} \simeq \mathrm{R\Gamma}(X_{\mathrm{dR}}, \mathscr{F}) \otimes_k k(\!(u)\!)$$

is an equivalence. In particular, taking $\mathscr{F} = \omega_{X_{\mathrm{dR}}}$ *we see that there is an equivalence*

$$\mathrm{R\Gamma}(LX, \omega_{LX})^{\mathrm{Tate}} \xrightarrow{\sim} \mathrm{R\Gamma}(X_{\mathrm{dR}}, \mathscr{F}) \otimes_k k(\!(u)\!) \simeq C^{dR,BM}_\bullet(X) \otimes_k k(\!(u)\!)$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} LX \arrow[r, "\pi"] \arrow[dr] & X_{\mathrm{dR}} \arrow[d, "p"] \\ & \mathrm{pt} \end{tikzcd}$$

Notice that each of the functors p_*, π_*, and $\pi^!$ are of the type discussed in Remark 4.6.3. The natural transformation

$$\pi_* \pi^! \mathscr{F} \longrightarrow \mathscr{F}$$

is a t-Tate-equivalence between left t-bounded objects by Theorem 5.1.3. Thus by Remark 4.6.3 the same is true of

$$\mathrm{R\Gamma}(X, \pi^! \mathscr{F}) \simeq \mathrm{R\Gamma}(X_{\mathrm{dR}}, \pi_* \pi^! \mathscr{F}) \longrightarrow \mathrm{R\Gamma}(X_{\mathrm{dR}}, \mathscr{F}).$$

We may now take the ordinary Tate construction to complete the proof. ☐

6.3. DCoh **and distributions on** LX.

6.3.1. Theorem 6.1.1 together with the "induction" functor $\operatorname{DCoh} X \to \operatorname{DCoh} X_{\mathrm{dR}}$ induce an $\mathrm{SO}(2)$-invariant trace

$$\mathbf{HH}_\bullet(\operatorname{DCoh} X) \longrightarrow \mathbf{HH}_\bullet(\operatorname{DCoh} X_{\mathrm{dR}}) \xrightarrow{\sim} C^{dR,BM}_\bullet(X)$$

We have the following

Theorem 6.3.2. *Suppose that X is a derived scheme locally of finite type. Then, the* $\mathrm{SO}(2)$*-equivariant trace above induces an equivalence*

$$\mathbf{HP}_\bullet\,(\operatorname{DCoh} X) \overset{\phi_X}{\simeq} C^{dR,BM}_\bullet(X)(\!(u)\!)$$

Proof. In order to compute $\mathbf{HH}_\bullet(\operatorname{DCoh} X)$, we follow a route similar to the above – only now we make it more systematic. We will construct a functor of symmetric monoidal ∞-categories

$$Z_{\operatorname{DCoh} X} \colon \mathrm{1Bord} \xrightarrow{X^{(-)}} \mathrm{Corr}' \xrightarrow{\mathrm{QC}^!(-)} \mathbf{dgcat}_k^\infty$$

where:

- 1Bord denotes the ∞-category of (unoriented) 1-bordisms. Objects are finite sets, 1-morphisms are (unoriented) cobordisms of finite sets, etc.
- Corr′ denotes the ∞-category of correspondences of derived schemes. In contrast to some variants, the only higher morphisms we allow are *equivalences* of correspodences. The left hand functor sends $[n] \mapsto X^n$ and a cobordism S between the finite sets $\delta_0 S$ and $\delta_1 S$ to the correspondence

$$X^{\delta_0 S} \leftarrow X^S \rightarrow X^{\delta_1 S}$$

- **$\mathbf{dgcat}_k^\infty$** denotes the ∞-category of presentable k-linear dg-categories and colimit preserving functors. The right hand functor sends a derived scheme X to $\mathrm{QC}^!(X)$ and a correspondence $X_0 \leftarrow X_{01} \rightarrow X_1$ to $(p_1)_*(p_0)^! \colon \mathrm{QC}^!(X_0) \to \mathrm{QC}^!(X_1)$.

Upcoming work of Gaitsgory-Rozenblyum [GR2] will show that $\mathrm{QC}^!$ has the desired functoriality for correspondences and is symmetric monoidal. Assuming this for now, let us complete the proof: It follows from the Cobordism Hypothesis that

$$Z_{\mathrm{DCoh}\, X}\left(S^1\right) : k\text{-mod} \to k\text{-mod}$$

is given by tensoring by $\mathbf{HH}_\bullet(\mathrm{DCoh}\, X)$. By the above description of Z, we see that

$$Z_{\mathrm{DCoh}\, X}\left(S^1\right) = \mathrm{R}\Gamma(LX, \omega_{LX} \otimes_k -) = \mathrm{R}\Gamma(LX, \omega_{LX}) \otimes_k -$$

with the SO(2)-action derived from that on LX and the SO(2)-equivariance of ω_{LX}: $\omega_{LX} \in \mathrm{QC}^!(LX)^{\mathrm{SO}(2)}$.

The analogous construction for $Z_{\mathrm{DCoh}\, X_{\mathrm{dR}}}$ factors canonically through the ordinary category Corr'_{var} of correspondence of *varieties*. In particular, it factors through the homotopy category of 1Bord, where the SO(2)-action on S^1 is necessarily trivial. There is a natural transformation $Z_{\mathrm{DCoh}\, X} \to Z_{\mathrm{DCoh}\, X_{\mathrm{dR}}}$ and we are studying its value on S^1.

We unwind this: Let $p_{dR} \colon X_{\mathrm{dR}} \to \mathrm{pt}$ and $\pi^{\mathrm{SO}(2)} \colon LX \to X_{\mathrm{dR}}$ the natural maps. Then, the natural map

$$\mathbf{HH}_\bullet(\mathrm{DCoh}\, X) = (p_{dR})_* \left(\pi_* \pi^! \omega_{X_{\mathrm{dR}}}\right) \longrightarrow (p_{dR})_* \omega_{X_{\mathrm{dR}}} = C_\bullet^{dR,BM}(X)$$

is SO(2)-equivariant for the trivial action on the right hand side. Prop. 6.2.1 proved that this induced an equivalence on Tate constructions. ☐

References

[BM] Kristine Bauer and Randy McCarthy, *On vanishing tate cohomology and decompositions in goodwillie calculus* (2003).

[BZNP] D. Ben-Zvi, D. Nadler, and A. Preygel, *Integral transforms for coherent sheaves* (2013), available at arxiv:1312.7164.

[DG] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, available at arxiv:1108.5351.

[FG] E. Frenkel and D. Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, Represent. Theory **13** (2009), 470–608, available at arxiv:0712.0788.

[G1] Dennis Gaitsgory, *Ind-coherent sheaves* (May 2011), available at arxiv:1105.4857.

[G2] Dennis Gaitsgory, *Functors given by kernels, adjunctions and duality* (2013), available at arxiv:1303.2763.

[GR1] D. Gaitsgory and N. Rozenblyum, *Crystals and d-modules*, available at arxiv:1111.2087.

[GR2] ______, *A study in derived algebraic geometry*, In preparation.

[GR3] ______, *Crystals and D-modules* (2011), available at arxiv:1111.2087.

[GR4] ______, *Dg Indschemes* (2011), available at arxiv:1108.1738.

[HH] John Harper and Kathryn Hess, *Homotopy completion and topological quillen homology of structured ring spectra*, Geometry & Topology, available at arxiv:1102.1234.

[HLP] Daniel Halpern-Leistner and Anatoly Preygel, *Mapping stacks and categorical notions of properness* (2014).

[L1] Jacob Lurie, *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems.*

[L2] ______, *Higher algebra.*

[P1] Anatoly Preygel, *Thom-Sebastiani & duality for matrix factorizations*, arXiv e-print (January 2011), available at arxiv:1101.5834.

[P2] ______, *Loop spaces and connections, revisted*, In preparation (2014).