A SPECTRAL INCARNATION OF AFFINE CHARACTER SHEAVES

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ABSTRACT. We present a Langlands dual realization of the putative category of affine character sheaves. Namely, we calculate the categorical center and trace (also known as the Drinfeld center and trace, or categorical Hochschild cohomology and homology) of the affine Hecke category starting from its spectral presentation. The resulting categories comprise coherent sheaves on the commuting stack of local systems on the two-torus satisfying prescribed support conditions, in particular singular support conditions as appear in recent advances in the Geometric Langlands program. The key technical tools in our arguments are: a new descent theory for coherent sheaves or $\mathcal{D}$-modules with prescribed singular support; and the theory of integral transforms for coherent sheaves developed in the companion paper [BZNP].

CONTENTS

1. INTRODUCTION

Let G be a complex reductive group with Langlands dual $G^\vee$. Thanks to Kazhdan-Lusztig [KL87], the affine Hecke algebra of $G^\vee$ admits a spectral description in terms of the K-group of equivariant coherent sheaves on the Steinberg variety of G, which results in a classification of irreducible representations (the Deligne-Langlands conjecture). Thanks to Bezrukavnikov [Bez], the affine Hecke category similarly admits a spectral description in terms of the category of equivariant coherent sheaves on the Steinberg variety, which one might hope to apply to describe the representation theory of the affine Hecke category. The main results of this paper are the calculation of the categorical center and trace of the affine Hecke category starting from

this spectral presentation. The resulting categories comprise coherent sheaves on the commuting stack, the derived stack of G-local systems on the two-torus, satisfying prescribed support conditions, in particular singular support conditions as appear in recent advances in the Geometric Langlands program [AG].

It is known that the categorical center [BFO12, BZNa] and trace [BZNa] of the finite Hecke category are equivalent to Lusztig's character sheaves. Thus one can view the results of this paper as giving a spectral construction of the putative category of affine character sheaves as the geometric Langlands category in genus one. The automorphic geometry of affine character sheaves continues to be the subject of much ongoing work motivated by representation theory of groups over local fields, with recent notable advances by Lusztig [Lusa, Lusb] and Bezrukavnikov, Kazhdan and Varshavsky [BKV] (the latter use the center of the affine Hecke category as a model for affine character sheaves). It is the natural home to a huge wealth of enumerative questions in representation theory and gauge theory (see for example [SV11, SV12, BZNb]).

Independently of specific applications, our proofs develop new descent techniques of broad applicability to coherent sheaves in derived algebraic geometry and $\mathcal{D}$-modules in microlocal geometry. In this introduction, we first explain the general techniques and then their specific application to the affine Hecke category. We conclude with a brief further discussion of the place of this work within geometric representation theory.

We will work throughout over an algebraically closed field k of characteristic zero. All constructions and terminology will refer to natural derived enhancements. For example, we will use the term category to stand for pre-triangulated k-linear dg category or stable k-linear ∞-category.

1.1. Singular support.

To any coherent $\mathcal{D}$-module $\mathcal{M}$ on a smooth scheme Z, one can associate a closed conic coisotropic subvariety $\mu(\mathcal{M}) \subset T^*Z$ called the singular support of $\mathcal{M}$. The intersection of the singular support with the zero-section is the traditional support of $\mathcal{M}$, and $\mathcal{M}$ is a vector bundle with flat connection if and only if the singular support lies in the zero-section. In general, the singular support records those codirections in which the propagation of sections of $\mathcal{M}$ are obstructed. In more traditional language, if one thinks of $\mathcal{M}$ as a generalized system of linear PDE, then the singular support comprises the wavefronts of distributional solutions.

Important categories of $\mathcal{D}$-modules are cut out by singular support conditions: holonomic $\mathcal{D}$-modules are those whose singular support is of minimal dimension and hence Lagrangian; Lusztig's character sheaves are adjoint-equivariant $\mathcal{D}$-modules on a reductive group with nilpotent singular support. As is familiar with linear PDE, many aspects of $\mathcal{D}$-modules, such as their classifications and functoriality, are best understood by viewing them microlocally via singular support.

Recent advances in the Geometric Langlands program [AG], building upon the study of categorical support in [BIK08], have brought sharpened attention to and deepened understanding of a parallel theory of singular support for coherent sheaves. We will continue by briefly highlighting some of the key ideas in this story with a further discussion to be found in Section 2.

The natural working context is now not a smooth scheme but a quasi-smooth derived scheme. Recall that a derived scheme Z is quasi-smooth if and only if it is a derived local complete intersection in the sense that it is Zariski-locally the derived zero-locus of a finite collection of polynomials. Equivalently, a derived scheme Z is quasi-smooth if and only if its cotangent complex $\mathbb{L}_Z$ is perfect of tor-amplitude $[-1, 0]$. More generally, it is possible to expand the working context to include derived stacks that are quasi-smooth in the sense that they admit a smooth atlas of quasi-smooth derived schemes.

To any quasi-smooth derived stack Z, with underlying classical stack Z^{cl}, one can attach its shifted cotangent bundle

$$T_Z^{*-1} = \operatorname{Spec}_{Z^{cl}} \operatorname{Sym}((\mathbb{L}_Z^{-1}[-1])^\vee)$$

The shifted cotangent bundle T_Z^{*-1} is a classical stack with a natural map $T_Z^{*-1} \to Z^{cl}$ with fibers the shift into degree 0 of the degree -1 cohomology of $\mathbb{L}_Z$. There is a natural closed embedding $Z^{cl} \subset T_Z^{*-1}$ of the zero-section which is an isomorphism over the smooth locus of Z.

Let $\operatorname{DCoh}(Z)$ denote the derived category of coherent complexes on Z, and $\operatorname{Perf}(Z)$ the derived category of perfect complexes. Functions on the shifted cotangent bundle T_Z^{*-1} naturally map to the graded center of the homotopy category of $\operatorname{DCoh}(Z)$. In this way, any coherent complex $\mathcal{M} \in \operatorname{DCoh}(Z)$ has a natural singular support $\mu(\mathcal{M}) \subset T_Z^{*-1}$ which is a closed conic subset. The singular support $\mu(M)$ records the failure of $\mathcal{M}$ to be a perfect complex: $\mu(\mathcal{M}) \cap Z^{cl}$ is the traditional support of $\mathcal{M}$, and $\mathcal{M} \in \operatorname{Perf}(Z)$ if and only if $\mu(\mathcal{M}) \subset Z^{cl}$. More precisely, the singular support $\mu(\mathcal{M})$ measures codirections of smoothings of Z in which

$\mathcal{M}$ is obstructed from extending as a coherent complex (see Remark 2.1.3). To any conic Zariski-closed subset $\Lambda \subset T_Z^{*-1}$, there is an intermediate small category

$$\mathrm{DCoh}(Z) \supset \mathrm{DCoh}_\Lambda(Z) \supset \mathrm{Perf}(Z)$$

consisting of coherent sheaves $\mathcal{M}$ with singular support $\mu(\mathcal{M}) \in \Lambda$.

Coherent complexes with interesting singular support arise from pushforward along proper but not smooth maps. A map of quasi-smooth stacks $f : W \to Z$ induces a correspondence

$$T_W^{*-1} \xleftarrow{f^*} T_Z^{*-1} \times_Z W \xrightarrow{\tilde{f}} T_Z^{*-1}$$

In analogy with traditional microlocal subsets, one can pushforward and pullback support conditions by taking them across the correspondence. In particular, one can measure the singularities of the map via its characteristic locus, the closed conic subset of covectors that pull back to the zero-section

$$\Lambda_f = \tilde{f}(W \times_{T_W^{*-1}} (T_Z^{*-1} \times_Z W)) \subset T_Z^{*-1}$$

Assuming f is proper, pushing forward perfect complexes along f produces coherent complexes with singularities in $\Lambda_f \subset T_Z^{*-1}$.

The appearance of singular support in this paper will result from studying descent along proper but not smooth maps. A main technical tool will be a new descent theory for coherent complexes with prescribed singular support. The arguments apply equally well in the more familiar setting of $\mathcal{D}$-modules and provide a new microlocal descent theory there as well.

1.2. **Convolution categories.** Next we introduce the general formalism of convolution categories of coherent complexes. We then state our main results about the calculations of their monoidal centers and traces.

Let $p : X \to Y$ be a proper map of derived stacks. We will ultimately apply this to simple concrete examples, but in general assume that X, Y are reasonable (derived Artin stacks over k of finite-presentation, geometric, and perfect in the sense of [BZFN10]) and that X is smooth.

In [BZNP], we prove general representability results for functors between categories of coherent sheaves as integral transforms with coherent kernels. In our present setting, we find that the integral transform construction provides a canonical equivalence

$$\Phi : \mathrm{DCoh}(X \times_Y X) \xrightarrow{\sim} \mathrm{Fun}^{ex}_{\mathrm{Perf}(Y)}(\mathrm{DCoh}\,X, \mathrm{DCoh}\,X) \qquad \Phi_{\mathcal{K}}(\mathcal{F}) = p_{Y*}(p_X^*(\mathcal{F}) \otimes \mathcal{K})$$

Here the functor category consists of exact $\mathrm{Perf}(Y)$-linear functors where $\mathrm{Perf}(Y)$ is monoidal and $\mathrm{DCoh}(X)$ is a module with respect to tensor product. Note that the functor category is naturally monoidal with respect to composition of functors and has a natural module $\mathrm{DCoh}(X)$.

Since X is smooth, the diagonal $\Delta : X \to X \times X$ has finite tor-dimension, so that convolution equips the category of integral kernels $\mathrm{DCoh}(X \times_Y X)$ with a natural monoidal structure

$$(X \times_Y X) \times (X \times_Y X) \xleftarrow{\delta_{23}} X \times_Y X \times_Y X \xrightarrow{\pi_{13}} X \times_Y X \qquad \mathcal{F}_1 * \mathcal{F}_2 = \pi_{13*}\delta_{23}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

Moreover, convolution also makes $\mathrm{DCoh}(X)$ into a natural $\mathrm{DCoh}(X \times_Y X)$-module

$$X \times (X \times_Y X) \xleftarrow{\delta_{12}} X \times_Y X \xrightarrow{\pi_3} X \qquad \mathcal{M} * \mathcal{F} = \pi_{3*}\delta_{12}^*(\mathcal{M} \boxtimes \mathcal{F})$$

We have the following basic compatibility.

Proposition 1.2.1. *The integral transform construction is naturally a monoidal equivalence*

$$\Phi : \mathrm{DCoh}(X \times_Y X) \xrightarrow{\sim} \mathrm{Fun}^{ex}_{\mathrm{Perf}(Y)}(\mathrm{DCoh}(X), \mathrm{DCoh}(X))$$

compatible with actions on the module $\mathrm{DCoh}(X)$.

Remark 1.2.2. One can also equip $\mathrm{DCoh}(X_1 \times_Y X_2)$ with the alternative !-convolution structure $\mathcal{F}_1 *^! \mathcal{F}_2 = \pi_{13*}\delta_{23}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. But tensoring with the pullback $p_2^*\omega_X$ of the dualizing complex intertwines the two monoidal structures. Likewise, tensoring with ω_X intertwines the two module structures on $\mathrm{DCoh}(X)$. In particular, if X is Calabi-Yau, the two monoidal structures coincide.

A natural challenge in geometric representation theory is to understand the module theory of the convolution category $\mathrm{DCoh}(X \times_Y X)$. It provides a highly structured version of the module theory of the affine Hecke algebra. In this paper, we will take the initial fundamental step and calculate its monoidal center and trace.

Definition 1.2.3. Let $\mathcal{A}$ be an algebra object in a symmetric monoidal ∞-category $\mathcal{C}$.

(1) The center (or Hochschild cohomology) is the morphism of bimodules object

$$Z(\mathcal{A}) = \mathrm{Hom}_{\mathcal{A}^{op} \otimes \mathcal{A}}(\mathcal{A}, \mathcal{A}) \in \mathcal{C}$$

It comes with a natural E_2-monoidal structure and universal central map $Z(\mathcal{A}) \to \mathcal{A}$.

(2) The trace (or Hochschild homology) is the tensor of bimodules object

$$Tr(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}^{op} \otimes \mathcal{A}} \mathcal{A} \in \mathcal{C}$$

It comes with a natural S^1-action and universal trace map $\mathcal{A} \to Tr(\mathcal{A})$.

Remark 1.2.4. We refer the reader to [Lur, 6.1,5.3] for the E_2-structure on the center (Deligne conjecture) and S^1-action on the trace (cyclic structure).

Remark 1.2.5. We will apply the above definitions to $\mathrm{DCoh}(X \times_Y X)$ considered as an algebra object in small stable categories. One could also pass to large categories and consider the cocompletion of ind-coherent sheaves $\mathrm{QC}^!(X \times_Y X) = \mathrm{Ind}\,\mathrm{DCoh}(X \times_Y X)$. The center is sensitive to the difference in context, while the trace is not: the trace of the cocompletion is canonically equivalent to the cocompletion of the trace.

The geometric avatar (or roughly speaking, E_∞-version) of the above definitions is the loop space.

Definition 1.2.6. The loop space of a derived stack Y is the derived mapping stack

$$LY = \mathrm{Map}(S^1, Y) \simeq Y \times_{Y \times Y} Y$$

Example 1.2.7. For G a group and $Y = BG$ the classifying stack, we have $LY \simeq G/G$ the adjoint quotient.

The geometric avatar of the universal central map and trace map is the correspondence

$$X \times_Y X \xleftarrow{\ \delta\ } (X \times_Y X) \times_{X \times X} X \simeq X \times_{Y \times X} X \xrightarrow{\ \pi\ } Y \times_{Y \times Y} Y \simeq LY$$

It corresponds to the cobordism with corners (the "whistle diagram" in topological field theory) between an interval with marked boundary and the circle.

Example 1.2.8. For $H \subset G$ a subgroup, and the natural map $X = BH \to Y = BG$, the correspondence becomes

$$H\backslash G/H \longleftarrow G/H \longrightarrow G/G$$

where the latter two terms are adjoint quotients.

It is shown in [BZFN10] that the resulting transforms $\pi_* \delta^*$ and $\delta_* \pi^*$ on quasicoherent sheaves induce respective equivalences

$$Tr(\mathrm{QC}(X \times_Y X)) \xrightarrow{\ \sim\ } \mathrm{QC}(LY) \xrightarrow{\ \sim\ } Z(\mathrm{QC}(X \times_Y X))$$

One can view our main results as a refinement for coherent sheaves in the presence of singularities. Let us first discuss the center where we need only impose traditional support conditions.

Definition 1.2.9. Let $\mathrm{Perf}_{prop/Y}(LY) \subset \mathrm{Perf}(LY)$ denote the full subcategory of perfect complexes that are proper over Y in the sense that their pushforward to Y is coherent.

An initial justification for the above definition is the fact that the functor $\delta_* \pi^* : \mathrm{QC}(LY) \to \mathrm{QC}(X \times_Y X)$ naturally restricts to a functor $\delta_* \pi^* : \mathrm{Perf}_{prop/Y}(LY) \to \mathrm{DCoh}(X \times_Y X)$.

The following theorem is our first main result. Its proof appeals to a substantial part of the theory of integral transforms for coherent sheaves developed in the companion paper [BZNP].

Theorem 1.2.10. *Suppose $p : X \to Y$ is a proper, surjective map of derived stacks with X, Y smooth.*

Then the functor $\delta_* \pi^* : \mathrm{Perf}_{prop/Y}(LY) \to \mathrm{DCoh}(X \times_Y X)$ is the universal central map underlying a canonical equivalence of E_2-monoidal categories

$$\mathrm{Perf}_{prop/Y}(LY) \xrightarrow{\ \sim\ } Z(\mathrm{DCoh}(X \times_Y X))$$

Now let us turn to the trace where we will need to consider singular support conditions. Define the support condition $\Lambda_{X/Y} \subset T^{*-1}_{LY}$ (see Section 2 for a precise discussion.) to be the the pull-push of support conditions

$$\Lambda_{X/Y} = p_* \delta^! T^{*-1}_{X \times_Y X}$$

Definition 1.2.11. Let $\mathrm{DCoh}_{\Lambda_{X/Y}}(LY) \subset \mathrm{DCoh}(LY)$ denote the full subcategory of coherent complexes whose microlocal support lies in $\Lambda_{X/Y} \subset T^{*-1}_{LY}$.

An initial justification for the above definition is the fact that the functor $\pi_* \delta^* : \mathrm{QC}(X \times_Y X) \to \mathrm{QC}(LY)$ naturally restricts to a functor $\pi_* \delta^* : \mathrm{DCoh}(X \times_Y X) \to \mathrm{DCoh}_{\Lambda_{X/Y}}(LY)$.

The following theorem is our second main result. Its proof appeals to the microlocal descent theory developed in this paper and outlined below in the next section of the introduction.

Theorem 1.2.12. *Suppose $p : X \to Y$ is a proper, surjective and quasi-smooth map of derived stacks with X, Y smooth.*

Then the functor $\pi_* \delta^* : \mathrm{DCoh}(X \times_Y X) \to \mathrm{DCoh}_{\Lambda_{X/Y}}(LY)$ is the universal trace map underlying a canonical equivalence of S^1-categories

$$Tr(\mathrm{DCoh}(X \times_Y X)) \xrightarrow{\ \sim\ } \mathrm{DCoh}_{\Lambda_{X/Y}}(LY)$$

Remark 1.2.13. As mentioned earlier, the trace is not sensitive to whether we pass to cocomplete categories: the trace of the cocompletion is canonically equivalent to the cocompletion of the trace. Thus the above theorem also implies the equivalence for ind-coherent sheaves

$$Tr(\mathrm{QC}^!(X \times_Y X)) \xrightarrow{\ \sim\ } \mathrm{QC}^!_{\Lambda_{X/Y}}(LY) = \mathrm{Ind}\,\mathrm{DCoh}_{\Lambda_{X/Y}}(LY)$$

1.3. **Base-change and descent with support.** Before continuing to applications, let us highlight the microlocal descent theory developed in Section 2 that contributes to the proof of Theorem 1.2.12. It is of independent interest and has broader applicability to $\mathcal{D}$-modules as well as coherent sheaves.

When working with fixed support conditions, natural functors on coherent sheaves need not respect the prescribed support conditions. For example, recall that perfect complexes are precisely coherent complexes with singular support in the zero-section. In general, pushforward of perfect complexes along a proper map takes perfect complexes to coherent complexes that are not perfect. If we insist on working with perfect complexes, then we must "correct" pushforwards so their singular support lies in the zero-section. One fallout is that standard identities such as base-change need not hold for such modified functors.

In Section 2, we introduce general geometric situations where base-change holds for functors with prescribed support conditions. This is a key step in establishing a general descent pattern (with respect to both pullback and pushforward) for coherent sheaves with prescribed support. A natural framework for such results is the geometry of pairs (X, Λ) of a quasi-smooth derived stack X and a conic Zariski-closed subset $\Lambda \subset T^{*-1}_X$. Morphisms are given by maps whose induced microlocal correspondences take the support condition of the domain to that of the target. To understand descent, we first derived a general form of base-change with prescribed support.

Definition 1.3.1. A *strict Cartesian diagram of pairs* is a Cartesian diagram of quasi-smooth derived stacks which is also a commutative diagram of maps of pairs

$$
\begin{array}{ccc}
(Z, \Lambda_Z) & \xrightarrow{\ p_2\ } & (X', \Lambda_{X'}) \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\
(X, \Lambda_X) & \xrightarrow{\ \ p\ \ } & (Y, \Lambda_Y)
\end{array}
$$

Furthermore, the pullbacks of support conditions should satisfy the strictness condition

$$\Lambda_Z \supset p_1^! \Lambda_X \cap p_2^! \Lambda_{X'}$$

Remark 1.3.2. Let us mention in a simple traditional setting the meaning of a map of pairs and what kind of notion strictness is. Take $f : X \to Y$ a smooth map of smooth manifolds, and consider the associated

Lagrangian correspondence

$$T^*X \xleftarrow{\ f^*\ } T_Y^* \times_Y X \xrightarrow{\ \tilde{f}\ } T^*Y$$

Fix support conditions $\Lambda_X \subset T^*X, \Lambda_Y \subset T^*Y$. Then f is a map of traditional pairs if the correspondence takes the support condition of the domain to that of the target: $f_*(\Lambda_X) = \tilde{f}((f^*)^{-1}(\Lambda_X)) \subset \Lambda_Y$. If f is a fibration, then f is a strict map of traditional pairs if the same additionally holds in the opposite direction: $f^!(\Lambda_Y) = f^*(\tilde{f}^{-1}(\Lambda_Y)) \subset \Lambda_X$.

In Prop. 2.3.7 and Prop. 2.3.8, we prove that for strict Cartesian diagrams of pairs with suitable properness and quasi-smoothness assumptions, both dual forms of base-change identities hold. These base-change identities allow us to prove descent theorems for both pullbacks and pushforwards by applying the Beck-Chevalley Condition [Lur, Corollary 6.2.4.3]. Given an augmented simplicial diagram $f\colon (X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1})$ of maps of pairs, we refer to the induced diagrams

$$
\begin{array}{ccc}
(X_{n+1}, \Lambda_{n+1}) & \xrightarrow{\ d_0\ } & (X_n, \Lambda_n) \\
{\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\
(X_{m+1}, \Lambda_{m+1}) & \xrightarrow{\ d_0\ } & (X_m, \Lambda_m)
\end{array}
$$

as the Beck-Chevalley squares. We then prove the following in Theorem 2.4.1.

Theorem 1.3.3. *Suppose $f\colon (X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1})$ is an augmented simplicial diagram of maps of pairs with all stacks quasi-smooth and maps proper. Suppose further that:*

    (i) The face maps are quasi-smooth.
    (ii) All Beck-Chevalley squares are strict Cartesian diagrams of pairs.
    (iii) $f_*\Lambda_0 = \Lambda_{-1}$.

Then the augmentation provides an equivalence with the totalization of the cosimplicial category furnished by pullbacks with support conditions:

$$\mathrm{QC}^!_{\Lambda_{-1}}(X_{-1}) \xrightarrow{\ \sim\ } \mathrm{Tot}\{\mathrm{QC}^!_{\Lambda_\bullet}(X_\bullet), f^!_\bullet\}$$

If in addition each of the $\mathrm{QC}^!_{\Lambda_k}(X_k)$ is compactly-generated for $k \geq 0$, then the same is true for $k = -1$, and the augmentation provides an equivalence with the geometric realization of the simplicial category furnished by pushforwards:

$$\mathrm{DCoh}_{\Lambda_{-1}}(X_{-1}) \xleftarrow{\ \sim\ } |\mathrm{DCoh}_{\Lambda_\bullet}(X_\bullet), f_{\bullet *}|$$

The arguments in the proof of the above theorem may be equally well implemented in the alternative setting of $\mathcal{D}$-modules. Namely, with analogous geometric hypotheses, the proof holds with $\mathcal{D}$-modules with prescribed singular support substituted for coherent sheaves with prescribed singular support. As far as we know, this is a new result going beyond the descent patterns appearing in [BD] and should have broad utility. For example, closely tied to the applications of this paper, it can be used to provide an alternative proof of one of the the main results of [BZNa] identifying the categorical trace of the finite Hecke category with character sheaves.

To state the version of the preceding theorem for $\mathcal{D}$-modules, we only need to change our microlocal setting back to the usual cotangent bundle. The natural framework is now the geometry of traditional pairs (X, Λ) of a smooth derived stack X and a conic Zariski-closed subset $\Lambda \subset T^*X$.

Let $\mathcal{D}_\Lambda(X)$ denote the full subcategory of the ind-completion of coherent $\mathcal{D}$-modules comprising objects with singular support lying in $\Lambda \subset T^*X$.

Theorem 1.3.4. *Suppose $f\colon (X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1})$ is an augmented simplicial diagram of maps of traditional pairs with all stacks smooth and maps proper. Suppose further that:*

    (i) The face maps are smooth.
    (ii) All Beck-Chevalley squares are strict Cartesian diagrams of pairs.
    (iii) $f^!\colon \mathcal{D}_{\Lambda_{-1}}(X_{-1}) \to \mathcal{D}_{\Lambda_0}(X_0)$ is conservative.

Then the augmentation provides an equivalence with the totalization of the cosimplicial category furnished by pullbacks with support conditions:

$$\mathcal{D}_{\Lambda_{-1}}(X_{-1}) \overset{\sim}{\longrightarrow} \mathrm{Tot}\{\mathcal{D}_{\Lambda_\bullet}(X_\bullet), \mathsf{f}_\bullet^!\}$$

Remark 1.3.5. In the descent theorem for coherent sheaves, we were able to give a criterion on support for the conservativity of pullback along the augmentation. We also were able to identify compact objects and give a pushforward formulation of descent on small categories. We are unsure if the analogous results hold for $\mathcal{D}$-modules in complete generality, though there are broad situations where they do.

1.4. **Application to affine Hecke categories.** Let us now turn to the motivating application for the development of the preceding theory.

Let G be a complex reductive group and $B \subset G$ a Borel subgroup. Let $q : BB \to BG$ denote the natural induction map of classifying stacks. Passing to loop spaces, we obtain the Grothendieck-Springer map of adjoint quotients

$$Lq : B/B \simeq \widetilde{G}/G \longrightarrow G/G$$

where $\widetilde{G}$ classifies pairs of a Borel subgroup $B' \subset G$ and a group element $g \in B'$, and Lq projects to the group element and forgets the Borel subgroup.

Now we will apply the preceding theory with $X = B/B$, $Y = G/G$, and $p = Lq$. Note that B/B and G/G are smooth, and $p : B/B \to G/G$ is projective. Note as well that our starting point already involves loop spaces, though that structure plays no role with respect to our general results.

Definition 1.4.1. (1) The global Steinberg stack is the fiber product

$$St_G = B/B \times_{G/G} B/B$$

(2) The global affine Hecke category is the small stable monoidal category

$$\mathcal{H}_G^{aff} = \mathrm{DCoh}(St_G)$$

Remark 1.4.2. One can interpret the loop space $L(BG) \simeq G/G$ as the moduli stack of G-local systems on the circle S^1. Similarly, one can interpret the global Steinberg stack $St_G \simeq L(B\backslash G/B)$ as the moduli of G-local systems on the cylinder $S^1 \times I$ with B-reductions at the boundary circles $S^1 \times \partial I$.

We will state the form our general results take when applied to the affine Hecke category.

Definition 1.4.3. The commuting stack is the moduli of local systems on the two-torus $T = S^1 \times S^1$, or equivalently, the twice-iterated loop space

$$\mathrm{Loc}_G(T) \simeq L(L(BG)) \simeq \{(g_1, g_2) \in G \times G \,|\, g_1 g_2 g_1^{-1} g_2^{-1} = 1\}/G$$

Remark 1.4.4. Unlike the Steinberg stack itself, the commuting stack has a nontrivial derived structure and must be treated as a derived stack.

Let $\mathfrak{g}$ denote the Lie algebra of G. The fiber of the cotangent complex of $\mathrm{Loc}_G(T)$ at a local system $\mathscr{P}$ can be calculated by the de Rham cochains $C^*(T, \mathfrak{g}_{\mathscr{P}}^*)[1]$, where $\mathfrak{g}_{\mathscr{P}}^*$ denotes the coadjoint bundle of $\mathscr{P}$. Focusing on the degree -1 term coming from the commutator equation, we see that there is a natural map

$$\mu : T^{*-1}_{\mathrm{Loc}_G(T)} \simeq C^0(T, \mathfrak{g}_{\mathscr{P}}^*) \longrightarrow \mathfrak{g}^*/G$$

Let $\mathfrak{h}$ denote the Lie algebra of the universal Cartan of G, and W the Weyl group. Recall the dual characteristic polynomial map, or equivalently, the projection to the coadjoint quotient

$$\chi : \mathfrak{g}^*/G \longrightarrow \mathfrak{g}^*/\!/G \simeq \mathfrak{h}^*/W$$

Define the global nilpotent cone $\mathcal{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$ to be the closed conic subset given by the inverse-image of zero under the composition

$$T^{*-1}_{\mathrm{Loc}_G(T)} \overset{\mu}{\longrightarrow} \mathfrak{g}^*/G \overset{\chi}{\longrightarrow} \mathfrak{h}^*/W$$

Definition 1.4.5. (1) Let $\mathrm{Perf}_{\mathscr{P}}(\mathrm{Loc}_G(T)) \subset \mathrm{Perf}(\mathrm{Loc}_G(T))$ denote the full subcategory of perfect complexes whose pushforward along the restriction map $\mathrm{Loc}_G(T) \to \mathrm{Loc}_G(S^1)$ along the first loop $S^1 \to T$ is coherent.

(2) Let $\mathrm{DCoh}_{\mathscr{N}}(\mathrm{Loc}_G(T)) \subset \mathrm{DCoh}(\mathrm{Loc}_G(T))$ denote the full subcategory of coherent complexes whose singular support lies in the global nilpotent cone $\mathscr{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$.

Theorem 1.4.6. *(1) There is a canonical monoidal equivalence*

$$\mathcal{H}_G^{aff} = \mathrm{DCoh}(St_G) \xrightarrow{\sim} \mathrm{Fun}^{ex}_{\mathrm{Perf}(G/G)}(\mathrm{Perf}(B/B), \mathrm{Perf}(B/B))$$

(2) There is a canonical E_2-monoidal identification of the center

$$\mathrm{Perf}_{\mathscr{P}}(\mathrm{Loc}_G(T)) \xrightarrow{\sim} Z(\mathcal{H}_G^{aff})$$

(3) There is a canonical S^1-equivariant identification of the trace

$$Tr(\mathcal{H}_G^{aff}) \xrightarrow{\sim} \mathrm{DCoh}_{\mathscr{N}}(\mathrm{Loc}_G(T))$$

Remark 1.4.7. Let us point out a particularly curious aspect of the theorem.

On the one hand, the description of the center is strongly asymmetric between the two loops of T. This is not surprising considering the two loops play different roles: the first is implicit in the adjoint quotients $L(BG) \simeq G/G, L(BB) \simeq B/B$ and hence in the global Steinberg stack as well $St_G = L(B\backslash G/B)$; the second arises in the geometric identification of the center.

On the other hand, the description of the trace is symmetric in the two loops.

Finally, our arguments also apply to more traditional versions of the affine Hecke category where we linearize and constrain our focus to nilpotent elements. Fix the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ of an invariant inner product. Let $\mathcal{B} = G/B$ denote the flag variety and $T^*\mathcal{B} \to \mathfrak{g}^* \simeq \mathfrak{g}$ the Springer/moment map.

Let us introduce the unipotent Steinberg stack

$$St_G^u = T^*\mathcal{B}/G \times_{\mathfrak{g}/G} T^*\mathcal{B}/G \simeq (T^*\mathcal{B}/G \times_{\mathfrak{g}} T^*\mathcal{B}/G)/G$$

Note that St_G^u has a nontrivial derived structure since we work over $\mathfrak{g}$ rather than the nilpotent cone. Note as well that we could equivalently work over the formal completion of $\mathfrak{g}$ along the nilpotent cone. Introduce the unipotent affine Hecke category

$$\mathcal{H}_G^{aff,u} = \mathrm{DCoh}(St_G^u)$$

and the unipotent commuting stack

$$\mathrm{Loc}_G(T)^u = \{(g_1, g_2) \in \hat{G}_u \times G \,|\, g_1 g_2 g_1^{-1} g_2^{-1} = 1\}/G$$

of local systems where the first monodromy $g_1 \in \hat{G}_u$ is in the formal neighborhood of the unipotent elements $G_u \subset G$. Now compatibly with [Theorem 1.4.6](#), our methods provide the following.

Theorem 1.4.8. *There are canonical identifications*

$$\mathrm{Perf}_{\mathscr{P}}(\mathrm{Loc}_G(T)^u) \xrightarrow{\sim} Z(\mathcal{H}_G^{aff,u}) \qquad Tr(\mathcal{H}_G^{aff,u}) \xrightarrow{\sim} \mathrm{DCoh}_{\mathscr{N}}(\mathrm{Loc}_G(T)^u)$$

which are E_2-monoidal and S^1-equivariant respectively.

Furthermore, one can introduce the natural $\mathbb{G}_m$-dilation action on $\mathfrak{g}$ and the induced action on St_G^u. Introduce the $\mathbb{G}_m$-equivariant unipotent affine Hecke category

$$\mathcal{H}_{G \times \mathbb{G}_m}^{aff,u} = \mathrm{DCoh}(St_G^u/\mathbb{G}_m)$$

and the twisted unipotent commuting stack

$$\mathrm{Loc}_G(T)^u_{\mathbb{G}_m} = \{(g_1, g_2, z) \in \hat{G}_u \times G \times \mathbb{G}_m \,|\, g_1 g_2 (g_1^{-1})^z g_2^{-1} = 1\}/(G \times \mathbb{G}_m)$$

where the first monodromy $g_1 \in \hat{G}_u$ is in the formal neighborhood of the unipotent elements $G_u \subset G$, and $(g_1^{-1})^z \in \hat{G}_u$ denotes the dilation of its inverse by the scalar $z \in \mathbb{G}_m$. Now our methods provide the following.

Theorem 1.4.9. *There are canonical identifications*

$$\operatorname{Perf}_{\mathscr{P}}(\operatorname{Loc}_G(T)^u_{\mathbb{G}_m}) \xrightarrow{\sim} Z(\mathcal{H}^{aff,u}_{G \times \mathbb{G}_m}) \qquad Tr(\mathcal{H}^{aff,u}_{G \times \mathbb{G}_m}) \xrightarrow{\sim} \operatorname{DCoh}_{\mathscr{N}}(\operatorname{Loc}_G(T)^u_{\mathbb{G}_m})$$

which are E_2-monoidal and S^1-equivariant respectively.

1.4.1. *Hecke categories, character sheaves and geometric Langlands.* We conclude this section with a brief, informal discussion of the place of Theorem 1.4.6, and its variants, within geometric representation theory. To match with the conventions of the subject, and for the purposes of this section only, the reductive group denoted above by G will be denoted $G^\vee$ since it will arise naturally as a Langlands dual group.

The unipotent Steinberg stack $St^u_{G^\vee}$ plays the central role in Kazhdan-Lusztig's solution [KL87] of the Deligne-Langlands conjecture on representations of affine Hecke algebras (see also [CG97]): the Grothendieck group of $\mathbb{G}_m$-equivariant coherent sheaves on $St^u_{G^\vee}$ is isomorphic to the affine Hecke algebra of G. This enables one to classify irreducible representations of the affine Hecke algebra in terms of q-commuting pairs.

Bezrukavnikov [Bez] has categorified the Kazhdan-Lusztig realization of the affine Hecke algebra: the standard categorification in terms of mixed sheaves on the affine flag variety of G is equivalent to the categorification by $\mathbb{G}_m$-equivariant coherent sheaves on $St^u_{G^\vee}$. The affine Hecke category appears naturally in the geometric Langlands program as the modifications acting on sheaves on moduli stacks of G-bundles with parabolic structure. Bezrukavnikov's theorem realizes the geometric Langlands duality for these modifications, or in other words the tamely ramified generalization of the geometric Satake theorem. It is the centerpiece in the geometric approach to a wide variety of problems in representation theory [Bez06].

With Geometric Langlands and other natural problems in mind, it is meaningful to study the representation theory of the affine Hecke category itself. By abstract nonsense, any dualizable module of a monoidal category has a character, which is an object in the trace category. Thus by Theorem 1.4.6 and its variants, characters of dualizable modules of the affine Hecke category give coherent sheaves with nilpotent singular support on the commuting stack $\operatorname{Loc}_{G^\vee}(T)$ and its unipotent and twisted variants. Note that this is consonant with the Deligne-Langlands classification of representations of the affine Hecke algebra in terms of q-commuting pairs. (This relation between the categorified and classical will be pursued in [BZHN14].)

It is also natural to relate Theorem 1.4.6 to the character theory of the finite Hecke category of Borel-biequivariant $\mathcal{D}$-modules on $G^\vee$. The main result of [BZNa] identifies the monoidal center and trace of the finite Hecke category with the category of unipotent character sheaves on $G^\vee$, that is, adjoint-equivariant $\mathcal{D}$-modules on $G^\vee$ with nilpotent singular support and trivial generalized central character. The relation between this and Theorem 1.4.6 is given by the results of [BZNc, BZNd]. Namely, coherent sheaves on a loop stack, such as the Steinberg stack $St_{G^\vee} = L(B^\vee \backslash G^\vee / B^\vee)$ or commuting stack $\operatorname{Loc}_{G^\vee}(T^2) = L(G^\vee / G^\vee)$, recover $\mathcal{D}$-modules on the stack via the process of S^1-localization and restriction to small loops. This supports the perspective that $\operatorname{DCoh}_{\mathscr{N}}(\operatorname{Loc}_G(T))$ is the spectral realization of the putative category of "affine character sheaves" for the p-adic group associated to G.

Finally, the trace category $\operatorname{DCoh}_{\mathscr{N}}(\operatorname{Loc}_{G^\vee}(T))$ is also closely related to the genus one case of the geometric Langlands conjecture. As formulated in [AG], the spectral side of the geometric Langlands conjecture on a smooth projective curve C is the category $\operatorname{DCoh}_{\mathscr{N}}(\operatorname{Conn}_{G^\vee}(C))$ of coherent sheaves with nilpotent singular support on the derived stack of flat $G^\vee$-connections on C. Note that the de Rham space $\operatorname{Conn}_{G^\vee}(C)$ can be identified analytically, though not algebraically, with the Betti space $\operatorname{Loc}_{G^\vee}(C)$. But unlike the de Rham space, the Betti space and hence the category $\operatorname{DCoh}_{\mathscr{N}}(\operatorname{Loc}_{G^\vee}(C))$ is a topological invariant of C. Thus the category $\operatorname{DCoh}_{\mathscr{N}}(\operatorname{Loc}_{G^\vee}(T))$ provides a topological version of the genus one geometric Langlands spectral category, and Theorem 1.4.6 ties it to the representation theory of the affine Hecke category.

1.5. **Standing assumptions.** Unless otherwise noted, our standing assumptions are as follows:

We work over a characteristic zero base field k.

By a *category* we will mean a k-linear stable dg-category or k-linear stable ∞-category.

By a *stack* $\mathscr{X}$, we will mean a derived Artin stack over k which is quasi-compact, almost of finite-presentation, and geometric. This implies that $\operatorname{QC}^!(\mathscr{X}) \simeq \operatorname{Ind}\operatorname{DCoh}(\mathscr{X})$ by [DG].

2. Base-change and descent with support

2.1. Preliminaries. We begin by collecting some basic notions about the singular support of coherent complexes (see [AG] for a comprehensive account).

2.1.1. *Odd cotangent bundle.* Let X be a quasi-smooth derived stack and $\mathbb{L}_X$ its cotangent complex. Let X_{cl} denote the underlying classical stack of X. Introduce the shifted cotangent complex

$$T_X^{*-1} = \operatorname{Spec}_{X_{cl}} \operatorname{Sym}_{X_{cl}} H^1(\mathbb{L}_X^\vee) \simeq (\operatorname{Spec}_X \operatorname{Sym}_X \mathbb{L}_X^\vee[1])_{cl}$$

There is a natural affine projection $T_X^{*-1} \to X_{cl}$ with fiberwise $\mathbb{G}_m$-action and the fiber $T_X^{*-1}|_x$ at a point $x \in X_{cl}$ is the degree -1 cohomology of $\mathbb{L}_Z|_x$. Informally, one can think of $T_X^{*-1} \to X_{cl}$ as a bundle of vector spaces of varying dimensions. We denote by $\{0\}_X \subset T_X^{*-1}$ the zero-section.

Example 2.1.1. If Z is a smooth scheme, and $X = LZ = \operatorname{Map}(S^1, Z)$ is its loop space, then $T_X^{*-1} \to X_{cl}$ is the usual cotangent bundle $T_Z^* \to Z$.

Example 2.1.2. If $X = BG$ is a classifying stack, then $T_X^{*-1} \simeq \mathfrak{g}^*/G \to BG$ is the coadjoint quotient.

2.1.2. *Microlocalization.* Let X be a quasi-smooth derived stack. Let $\operatorname{Con} X$ denote the set of arbitrary unions of closed conic subsets of T_X^{*-1}. An important invariant of any $\mathscr{F} \in \operatorname{QC}^!(X)$ is its singular support

$$\operatorname{supp} \mathscr{F} \subset T_X^{*-1}$$

It is a conic Zariski-closed subset when $\mathscr{F} \in \operatorname{DCoh}(X)$ and in general a union of conic Zariski-closed subsets.

Singular support is a smooth-local notion and given by the following construction for X affine (see also the description of Remark 2.1.3). There is a natural map of graded commutative $\mathcal{O}(X)$-algebras

$$\mathcal{O}(T_X^{*-1}) \longrightarrow HH^{ev}(X)$$

to the even Hochschild cohomology restricting to maps

$$\mathcal{O}(X_{cl}) \longrightarrow HH^0(X) \qquad H^1(\mathbb{L}_X^\vee) \longrightarrow HH^2(X)$$

In turn, there is natural map from $HH^{ev}(X)$ to the graded center of the homotopy category of $\operatorname{QC}^!(X)$. The singular support $\operatorname{supp} \mathscr{F} \subset T_X^{*-1}$ of an object $\mathscr{F} \in \operatorname{QC}^!(X)$ is its traditional support under the induced central action of $\mathcal{O}(T_X^{*-1})$.

Let $\operatorname{Con} X$ denote the set of conic Zariski-closed subsets of T_X^{*-1}. For any $\Lambda \in \operatorname{Con} X$, one defines the full subcategory

$$i_\Lambda \colon \operatorname{QC}^!_\Lambda(X) \lhook\joinrel\longrightarrow \operatorname{QC}^!(X)$$

of ind-coherent complexes supported along Λ. The inclusion i_Λ admits a right adjoint

$$\operatorname{R\Gamma}_\Lambda : \operatorname{QC}^!(X) \longrightarrow \operatorname{QC}^!_\Lambda(X)$$

We will often regard $\operatorname{QC}^!_\Lambda(X)$ as a subcategory of $\operatorname{QC}^!(X)$ via the embedding i_Λ, and also regard $\operatorname{R\Gamma}_\Lambda$ as an endofunctor of $\operatorname{QC}^!(X)$.

We set $\operatorname{DCoh}_\Lambda(X) = \operatorname{DCoh}(X) \cap \operatorname{QC}^!_\Lambda(X)$. By [AG, Cor. 8.2.8], for global complete intersection stacks (in the sense of [AG, Sect. 8.2]), we have $\operatorname{QC}^!_\Lambda(X) = \operatorname{Ind} \operatorname{DCoh}_\Lambda(X)$.

Remark 2.1.3. For $\mathscr{F} \in \operatorname{DCoh} X$, one has $\operatorname{supp} \mathscr{F} \subset \{0\}_X$ if and only if $\mathscr{F} \in \operatorname{Perf} X$. This observation can be upgraded to a geometric description of $\operatorname{supp} \mathscr{F}$ as follows.

Suppose that $\eta \colon \operatorname{Spec} k \to X$ is a geometric point, and that in a neighborhood of η, one has a presentation of X as an iterated fiber

$$
\begin{array}{ccccc}
X & \lhook\joinrel\longrightarrow & X' & \lhook\joinrel\longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f_1,\ldots,f_n} \\
\{0 \in \mathbb{A}^n\} & \lhook\joinrel\longrightarrow & \mathbb{A}^1 \times \{0 \in \mathbb{A}^{n-1}\} & \lhook\joinrel\longrightarrow & \mathbb{A}^n
\end{array}
$$

where M is affine and smooth. Then one can interpret df_1 as a section of $T^{*-1}X$ and one has

$$df_1|_\eta \notin \operatorname{supp} \mathscr{F}|_\eta \subset T^{*-1}X\big|_\eta$$

if and only if $\mathscr{F}$ is contained in the smallest thick subcategory of $\operatorname{DCoh} X$ generated by pullbacks from $\operatorname{DCoh} X'$. Informally speaking, this is the case when "$\mathscr{F}$ extends in the f_1 direction near η".

Lemma 2.1.4. *For $\Lambda, \Lambda' \in \operatorname{Con} X$, there is a natural equivalence $\operatorname{R}\Gamma_\Lambda \circ \operatorname{R}\Gamma_{\Lambda'} \simeq \operatorname{R}\Gamma_{\Lambda \cap \Lambda'}$.*

Proof. See [AG, Prop. 2.2.6]. $\qquad\square$

2.1.3. *Functoriality.* Associated to a map $f: X \to Y$ is a correspondence

$$\text{(1)} \qquad\qquad T_X^{*-1} \xleftarrow{\ df^*\ } T_Y^{*-1} \times_Y X \xrightarrow{\ \tilde{f}\ } T_Y^{*-1}$$

Definition 2.1.5. Let $f: X \to Y$ be a map of quasi-smooth stacks.

(i) Given a subset $U \subset T_X^{*-1}$, we may form the subset
$$f_* U = \tilde{f}((df^*)^{-1}(U)) \subset T_Y^{*-1}$$
If $f: X \to Y$ is proper, then $\tilde{f}$ is proper, and this defines a map
$$f_*: \operatorname{Con} X \longrightarrow \operatorname{Con} Y$$

(ii) Given a subset $V \subset T_Y^{*-1}$, we may form the subset
$$f^! V = df^*(X \times_Y V) \subset T_X^{*-1}$$
If $f: X \to Y$ is quasi-smooth, then df^* is a closed immersion, and this defines a map
$$f^!: \operatorname{Con} Y \longrightarrow \operatorname{Con} X$$

Lemma 2.1.6. *Let $f: X \to Y$ be a map of quasi-smooth stacks.*

(i) *Suppose that $\mathscr{F} \in \operatorname{QC}^!(X)$ and that f is schematic and quasi-compact. Then, we have*
$$\operatorname{supp} f_* \mathscr{F} \subset f_* \operatorname{supp} \mathscr{F}$$
Thus if $\tilde{f}_* \Lambda_X \subset \Lambda_Y$, then
$$f_*(\operatorname{QC}^!_{\Lambda_X}(X)) \subset \operatorname{QC}^!_{\Lambda_Y}(Y)$$

(ii) *Suppose that $\mathscr{F} \in \operatorname{QC}^!(Y)$. Then, we have*
$$\operatorname{supp} f^! \mathscr{F} \subset f^! \operatorname{supp} \mathscr{F}$$
Thus if $f^! \Lambda_Y \subset \Lambda_X$, then
$$f^!(\operatorname{QC}^!_{\Lambda_Y}(Y)) \subset \operatorname{QC}^!_{\Lambda_X}(X)$$

Proof. See [AG, Lemma 7.4.5] for pushforwards and [AG, Lemma 7.4.2] for pullbacks. $\qquad\square$

We have the following partial converse to Lemma 2.1.6.

Proposition 2.1.7. *Let $f: X \to Y$ be a map of quasi-smooth stacks.*

(i) *Suppose that f is schematic and proper. Then $\operatorname{QC}^!_{f_* \Lambda_X}(X)$ is the smallest full subcategory of $\operatorname{QC}^!(Y)$ containing the essential image $f_*(\operatorname{QC}^!_{\Lambda_X}(X))$ and closed under colimits.*

(ii) *Suppose that f is quasi-smooth. Then $\operatorname{QC}^!_{f^! \Lambda_Y}(X)$ is the smallest full subcategory of $\operatorname{QC}^!(X)$ containing the essential image $f^!(\operatorname{QC}^!_{\Lambda_Y}(Y))$ and closed under colimits and tensoring by objects of $\operatorname{QC}(X)$.*

Proof. See [AG, Prop. 7.4.19] for assertion (i) and [AG, Prop. 7.4.14] for asserion (ii). $\qquad\square$

Remark 2.1.8. The proposition implies that for $\mathscr{F}' \in \operatorname{DCoh}(X')$ (resp., $\mathscr{F} \in \operatorname{DCoh}(X)$) the singular support $\operatorname{supp} f_* \mathscr{F} \in \operatorname{Con} X$ (resp., $\operatorname{supp} f^! \mathscr{F}' = \operatorname{supp} f^* \mathscr{F}' \in \operatorname{Con} X'$) depends only on $\operatorname{supp} \mathscr{F}' \in \operatorname{Con} X'$ (resp., $\operatorname{supp} \mathscr{F} \in \operatorname{Con} X$).

2.1.4. *Relative tensor products.*

Proposition 2.1.9. *Let X_1, X_2 be quasi-smooth stacks over a smooth separated base Y. Then the functor of exterior product over Y induces an equivalence*

$$\mathrm{DCoh}(X_1) \otimes_{\mathrm{Perf}(Y)} \mathrm{DCoh}(X_2) \xrightarrow{\boxtimes_Y} \mathrm{DCoh}(X_1 \times_Y X_2)$$

$$\xrightarrow{\sim} \mathrm{DCoh}_\Lambda(X_1 \times_Y X_2)$$

where $\Lambda = i^!(T^{*-1}_{X_1 \times X_2})$ for $i : X_1 \times_Y X_2 \to X_1 \times X_2$.

Proof. Recall by [Pre11, Proposition B.1.1] that $\boxtimes_Y$ is fully faithful. By [Pre11, Theorem B.2.4] exterior products over k generate $\mathrm{DCoh}(X_1 \times X_2)$. Observe that i is an affine quasi-smooth morphism between quasi-smooth stacks, since it is a base-change of the diagonal of Y. Thus the proof of [AG, Prop. 7.4.12] implies that the essential image of i^* generates $\mathrm{DCoh}_Y(X_1 \times_Y X_2)$. ☐

2.2. Maps of pairs.

Definition 2.2.1. Let X, Y be quasi-smooth stacks, and $\Lambda_X \in \mathrm{Con}\, X, \Lambda_Y \in \mathrm{Con}\, Y$.
   Define a *map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$* to be a map $f : X \to Y$ such that $f_* \Lambda_X \subset \Lambda_Y$.
   In this case, we say "f takes Λ_X to Λ_Y".

Remark 2.2.2. Returning to the correspondence (Equation 1), let us spell out the above definition.
   For a map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$, we require

$$(df^*)^{-1}(\Lambda_X) \subset X \times_Y \Lambda_Y$$

If $f : X \to Y$ is quasi-smooth, so that df^* is a closed immersion, then we can equivalently require

$$df^*(X \times_Y T^{*-1}_Y) \cap \Lambda_X \subset df^*(X \times_Y \Lambda_Y)$$

With our previous notation, this can be rephrased in the form

$$f^! T^{*-1}_Y \cap \Lambda_X \subset f^! \Lambda_Y$$

Lemma 2.2.3. *If $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a map of pairs, then f_* takes $\mathrm{R}\Gamma_{\Lambda_X}$-local objects to $\mathrm{R}\Gamma_{\Lambda_Y}$-local objects. If f is proper and quasi-smooth, then f provides a map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ if and only if $f_* \mathrm{DCoh}_{\Lambda_X}(X) \subset \mathrm{DCoh}_{\Lambda_Y}(Y)$.*

Proof. Both assertions are immediate from Lemma 2.1.6. ☐

Definition 2.2.4. Let X, Y be quasi-smooth stacks, and $\Lambda_X \in \mathrm{Con}\, X, \Lambda_Y \in \mathrm{Con}\, Y$.
   Define a *strict map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$* to be a map $f : X \to Y$ such that

$$(df^*)^{-1}(\Lambda_X) = X \times_Y \Lambda_Y$$

In this case, we say "the f-preimage of Λ_Y is precisely Λ_X".

Remark 2.2.5. If $f : X \to Y$ is quasi-smooth, so that df^* is a closed immersion, then $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a strict map of pairs if and only if

$$df^*(X \times_Y T^{*-1}_Y) \cap \Lambda_X = df^*(X \times_Y \Lambda_Y)$$

With our previous notation, this can be rephrased in the form

$$f^! T^{*-1}_Y \cap \Lambda_X = f^! \Lambda_Y$$

In practice, the above definition is too restrictive.

Definition 2.2.6. Let X, Y be quasi-smooth stacks, and $\Lambda_X, \Lambda'_X \in \mathrm{Con}\, X, \Lambda_Y \in \mathrm{Con}\, Y$.
   A map $f : X \to Y$ is said to be a *strict map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ along Λ'_X* if we have

$$(df^*)^{-1}(\Lambda_X \cap \Lambda'_X) = (df^*)^{-1}(\Lambda'_X) \cap (X \times_Y \Lambda_Y)$$

In this case, we say "along Λ'_X, the f-preimage of Λ_Y is precisely Λ_X".

Remark 2.2.7. If $f \colon X \to Y$ is quasi-smooth, so that df^* is a closed immersion, then $f \colon (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a strict map of pairs along Λ'_X if and only if

$$df^*(X \times_Y T_Y^{*-1}) \cap \Lambda_X \cap \Lambda'_X = df^*(X \times_Y \Lambda_Y) \cap \Lambda'_X$$

With our previous notation, this can be rephrased in the form

$$f^! T_Y^{*-1} \cap \Lambda_X \cap \Lambda'_X = f^! \Lambda_Y \cap \Lambda'_X$$

If in addition $f \colon X \to Y$ is already known to be a map of pairs $f \colon (X, \Lambda_X) \to (Y, \Lambda_Y)$, so that

$$f^! T_Y^{*-1} \cap \Lambda_X = df^*(X \times_Y T_Y^{*-1}) \cap \Lambda_X \subset df^*(X \times_Y \Lambda_Y) = f^! \Lambda_Y$$

then it is strict along Λ'_X if and only if

$$\Lambda_X \supset df^*(X \times_Y \Lambda_Y) \cap \Lambda'_X = f^! \Lambda_Y \cap \Lambda'_X$$

2.3. **Base-change with support.**

Lemma 2.3.1. *Let X, Y be quasi-smooth stacks, and $\Lambda_X, \Lambda'_X \in \mathrm{Con}\, X, \Lambda_Y \in \mathrm{Con}\, Y$.*
Suppose that $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a quasi-smooth map of pairs.
Then there is a natural morphism

$$f_* \circ R\Gamma_{\Lambda_X} \longrightarrow R\Gamma_{\Lambda_Y} \circ f_*$$

of functors $\mathrm{QC}^!(X) \to \mathrm{QC}^!_{\Lambda_Y}(Y)$.
Furthermore, if $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is strict along Λ'_X, then the above morphism is an equivalence when restricted to the full subcategory

$$\mathrm{QC}^!_{\Lambda'_X}(X) \subset \mathrm{QC}^!(X)$$

Proof. First, from the counit $i_{\Lambda_X} \circ R\Gamma_{\Lambda_X} \to 1$, we obtain a map $f_* \circ R\Gamma_{\Lambda_X} \to f_*$. Since f is a map of pairs, $f_* \circ R\Gamma_{\Lambda_X}$ lands in the $R\Gamma_{\Lambda_Y}$-local objects. Thus $f_* \circ R\Gamma_{\Lambda_X} \to f_*$ factors through $R\Gamma_{\Lambda_Y} \circ f_*$.

Now assume f is strict along Λ'_X. We must show that the map

$$f_* \circ R\Gamma_{\Lambda_X} \longrightarrow R\Gamma_{\Lambda_Y} \circ f_*$$

is an equivalence on $\mathrm{QC}^!_{\Lambda'_X}(X)$.

Suppose $\mathscr{F} \in \mathrm{QC}^!_{\Lambda'_X}(X)$. We must show that the natural map

$$f_* \circ R\Gamma_{\Lambda_X} \mathscr{F} \longrightarrow f_* \mathscr{F}$$

is an $R\Gamma_{\Lambda_Y}$-equivalence. Equivalently, we must show that the induced map

$$\mathrm{RHom}_{\mathrm{QC}^!(Y)}(\mathscr{K}, f_* R\Gamma_{\Lambda_X} \mathscr{F}) \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(Y)}(\mathscr{K}, f_* \mathscr{F})$$

is an equivalence for all $\mathscr{K} \in \mathrm{DCoh}_{\Lambda_Y}(Y)$.

Since f is quasi-smooth, f^* exists and is left adjoint to f_*, so the above is equivalent to showing that

$$\mathrm{RHom}_{\mathrm{QC}^!(Y)}(f^* \mathscr{K}, R\Gamma_{\Lambda_X} \mathscr{F}) \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(Y)}(f^* \mathscr{K}, \mathscr{F})$$

is an equivalence.

By Lemma 2.1.6 and the comparison between f^* and $f^!$ for quasi-smooth maps (thanks to the fact that f is quasi-smooth hence Gorenstein), we have $f^* \mathscr{K} \in \mathrm{QC}^!_{f^! \Lambda_Y}(X)$, and thus the above map is equivalent to a map

$$\mathrm{RHom}_{\mathrm{QC}^!(Y)}(f^* \mathscr{K}, R\Gamma_{f^! \Lambda_Y} R\Gamma_{\Lambda_X} \mathscr{F}) \longrightarrow \mathrm{RHom}_{\mathrm{QC}^!(Y)}(f^* \mathscr{K}, R\Gamma_{f^! \Lambda_Y} \mathscr{F})$$

Finally, since $\mathscr{F}$ is already $R\Gamma_{\Lambda'_X}$-local, by Lemma 2.1.4, we have that

$$R\Gamma_{f^! \Lambda_Y} R\Gamma_{\Lambda_X} \mathscr{F} \simeq R\Gamma_{f^! \Lambda_Y} R\Gamma_{\Lambda_X} R\Gamma_{\Lambda'_X} \mathscr{F} \simeq R\Gamma_{f^! \Lambda_Y \cap \Lambda_X \cap \Lambda'_X} \mathscr{F}$$

The strictness of f along Λ'_X precisely guarantees that

$$f^! \Lambda_Y \cap \Lambda_X \cap \Lambda'_X = f^! \Lambda_Y \cap \Lambda'_X \qquad \qquad \square$$

Definition 2.3.2. Suppose $f \colon X \to Y$ is a map of quasi-smooth stacks.

Fix $\Lambda_X \in \operatorname{Con} X$, $\Lambda_Y \in \operatorname{Con} Y$, and define *functors with support conditions*

$$\mathfrak{f}_* \colon \operatorname{QC}^!_{\Lambda_X}(X) \longrightarrow \operatorname{QC}^!_{\Lambda_Y}(Y) \qquad \mathfrak{f}^! \colon \operatorname{QC}^!_{\Lambda_Y}(Y) \longrightarrow \operatorname{QC}^!_{\Lambda_X}(X)$$

$$\mathfrak{f}_* = R\Gamma_{\Lambda_Y} \circ f_* \circ i_{\Lambda_X} \qquad \mathfrak{f}^! = R\Gamma_{\Lambda_X} \circ f^! \circ i_{\Lambda_Y}$$

Remark 2.3.3. If $f \colon (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a map of pairs, then $f_* \operatorname{QC}^!_{\Lambda_X}(X) \subset \operatorname{QC}^!_{\Lambda_Y}(Y)$, so we need not apply $R\Gamma_{\Lambda_Y}$ in the definition of $\mathfrak{f}_*$, or in other words $\mathfrak{f}_* \simeq f_* \circ i_{\Lambda_X}$, and hence $\mathfrak{f}_*$ preserves compact objects. Thus if in addition f is proper, the right adjoint to $\mathfrak{f}_* \simeq f_* \circ i_{\Lambda_X}$ coincides with $\mathfrak{f}^!$. (Note that we still must apply $R\Gamma_\Lambda$ in the definition of $\mathfrak{f}^!$ in general: if f is proper but a support condition is not satisfied, then $f^!$ need not be right adjoint to $\mathfrak{f}_*$.[1])

Similarly, suppose $f \colon (X, \Lambda_X) \to (Y, \Lambda_Y)$ and $g \colon (Y, \Lambda_Y) \to (Z, \Lambda_Z)$ are proper maps of pairs. Then $h = g \circ f$ is also a proper map of pairs, and there is a natural equivalence $\mathfrak{h}_* \simeq \mathfrak{g}_* \circ \mathfrak{f}_*$. By the above discussion, this follows from functoriality of the usual pushforwards. Moreover, taking right adjoints, we obtain a natural equivalence $\mathfrak{h}^! \simeq \mathfrak{f}^! \circ \mathfrak{g}^!$.

It will be crucial for us to study "base-change" for these functors with support conditions. First, we recall the following general context for discussing base change equivalences from [Lur, 6.2.3.13].

Definition 2.3.4. Suppose given a diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;U\;} & \mathcal{D} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G'} \\ \mathcal{C}' & \xrightarrow{\;V\;} & \mathcal{D}' \end{array}$$

which commutes up to a specified equivalence

$$\alpha \colon V \circ G \xrightarrow{\;\sim\;} G' \circ U$$

(1) We say that the square is *left adjointable* if the functors G and G' admit left adjoints F and F', and base-change holds: the composite transformation

$$F' \circ V \xrightarrow{\;\eta\;} F' \circ V \circ G \circ F \xrightarrow{\;\alpha\;} F' \circ G' \circ U \circ F \xrightarrow{\;\epsilon\;} U \circ F$$

is an equivalence, where η and ϵ are the respective unit and counit of adjunctions.

(2) Dually, the square is *right adjointable* if the functors G and G' admit right adjoints H and H', and the composite transformation

$$U \circ H \xrightarrow{\;\eta\;} H' \circ G' \circ U \circ H \xrightarrow{\;\alpha^{-1}\;} H' \circ V \circ G \circ H \xrightarrow{\;\epsilon\;} H' \circ V$$

is an equivalence, where η and ϵ are the respective unit and counit of adjunctions.

Definition 2.3.5. A *strict Cartesian diagram of pairs* is a Cartesian diagram of quasi-smooth stacks which is also a commutative diagram of maps of pairs

$$\begin{array}{ccc} (Z = X \times_S X', \Lambda_Z) & \xrightarrow{\;p_2\;} & (X', \Lambda_{X'}) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\ (X, \Lambda_X) & \xrightarrow{\;p\;} & (Y, \Lambda_Y) \end{array}$$

satisfying the strictness condition

$$\Lambda_Z \supset p_1^! \Lambda_X \cap p_2^! \Lambda_{X'}$$

---

[1]Example: Let $f \colon X = \operatorname{Spec} k \to Y = \Omega_0 \mathbb{A}^1 = \operatorname{Spec} k[B]$, with $|B| = 1$. Set $\Lambda_Y = \{0\}_Y$. Then $f_* \colon k\text{-mod} \to k[B]\text{-mod}$ is the usual pushforward with right adjoint $f^! = \operatorname{RHom}_{k[B]}(k, -)$. However $\mathfrak{f}^!$ is the colimit-preserving functor $\operatorname{RHom}_{k[B]}(k, k[B]) \otimes_{k[B]} -$.↑

Remark 2.3.6. If p_1 and p_2 are in addition assumed quasi-smooth, then by Remark 2.2.7 the strictness condition $\Lambda_Z \supset p_1^! \Lambda_X \cap p_2^! \Lambda_{X'}$ above is equivalent to any of the following:

- p_1 is strict along $p_2^! \Lambda_{X'}$;
- p_2 is strict along $p_1^! \Lambda_X$.

Proper base-change can be worded as a right adjointability condition.

Proposition 2.3.7. *Consider a strict Cartesian diagram of pairs*

$$
\begin{array}{ccc}
(Z = X \times_S X', \Lambda_Z) & \xrightarrow{\;p_2\;} & (X', \Lambda_{X'}) \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
(X, \Lambda_X) & \xrightarrow[\;p\;]{} & (Y, \Lambda_Y)
\end{array}
$$

Assume:

- *p is proper (and consequently so is p_2).*
- *p_1 is quasi-smooth.*

Then:

(i) We have adjunctions

$$
(\mathfrak{p}_* = R\Gamma_{\Lambda_Y} \circ p_*, \mathfrak{p}^! = R\Gamma_{\Lambda_X} \circ p^!) \qquad (\mathfrak{p}_{2*} = R\Gamma_{\Lambda_{X'}} \circ p_{2*}, \mathfrak{p}_2^! = R\Gamma_{\Lambda_Z} \circ p_2^!)
$$

(ii) The diagram of pushforwards

$$
\begin{array}{ccc}
\mathrm{QC}^!_{\Lambda_Z}(Z) & \xrightarrow{\;\mathfrak{p}_{2*}\;} & \mathrm{QC}^!_{\Lambda_{X'}}(X') \\
{\scriptstyle \mathfrak{p}_{1*}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{q}_*} \\
\mathrm{QC}^!_{\Lambda_X}(X) & \xrightarrow[\;\mathfrak{p}_*\;]{} & \mathrm{QC}^!_{\Lambda_Y}(Y)
\end{array}
$$

admits a natural equivalence

$$
\mathfrak{p}_* \circ \mathfrak{p}_{1*} \simeq \mathfrak{p}_{2*} \circ \mathfrak{q}_*
$$

and is right adjointable: the resulting base-change morphism

$$
\mathfrak{p}_{1*}\mathfrak{p}_2^! \longrightarrow \mathfrak{p}^! \circ \mathfrak{q}_*
$$

is an equivalence.

Proof. Point (i) and the functoriality equivalence of point (ii) are immediate from Remark 2.3.3. The adjointability morphism of point (ii) is the composite

$$
\mathfrak{p}_{1*} \circ \mathfrak{p}_2^! \simeq p_{1*} \circ R\Gamma_{\Lambda_Z} \circ p_2^! \longrightarrow R\Gamma_{\Lambda_X} \circ p_{1*} \circ p_2^! \longrightarrow R\Gamma_{\Lambda_X} \circ p^! \circ q_* \simeq \mathfrak{p}^! \circ \mathfrak{q}_*
$$

Note that the second arrow is an equivalence by the usual base-change theorem for $\mathrm{QC}^!$. To see that the first arrow is an equivalence we apply Lemma 2.3.1 as follows:

Note that the essential image of $p_2^!$ on $\mathrm{QC}^!_{\Lambda_{X'}}(X')$ lies in $\mathrm{QC}^!_{p_2^!\Lambda_{X'}}(Z)$ by Lemma 2.1.6. Since p_2 is not assumed quasi-smooth, $p_2^!\Lambda_{X'}$ need not be closed and by $\mathrm{QC}^!_{p_2^!\Lambda_{X'}}(Z)$ we mean the subcategory of $\mathrm{QC}^!(Z)$ generated under colimits by all coherent complexes on Z whose microsupport is contained is a conical closed subset contained in $p_2^!\Lambda_{X'}$. Thus it is enough to show that the natural morphism

$$
p_{1*} \circ R\Gamma_{\Lambda Z} \longrightarrow R\Gamma_{\Lambda_X} \circ p_{1*}
$$

is an equivalence on $\mathrm{QC}^!_{\Lambda'_Z}(Z)$ for each $\Lambda'_Z \in \mathrm{Con}\, Z$ contained in $p_2^!\Lambda_{X'}$.

Since the diagam is a strict Cartesian diagram of pairs and p_1 is quasi-smooth, Remark 2.2.7 implies that p_1 is strict along $p_2^!\Lambda_{X'}$ and thus along each such Λ'_Z. Lemma 2.3.1 now completes the proof.[2] $\qquad\square$

---

[2]The reader can note that we needed slightly less than a strict Cartesian diagram. The strictness is equivalent to p_1 being strict along $p_2^!\Lambda_{X'}$, while we needed it only along the union of all conical closed subsets of $p_2^!\Lambda_{X'}$. If, as in our examples, p_2 is also quasi-smooth then this distinction disappears.↑

With slightly more stringent conditions, we can interpret the (dual) base-change equivalence as an adjointability statement for the diagram of !-pullbacks (instead of pushfowards).

Proposition 2.3.8. *Consider a strict Cartesian diagram of pairs*

$$
\begin{array}{ccc}
(Z = X \times_S X', \Lambda_Z) & \xrightarrow{\;p_2\;} & (X', \Lambda_{X'}) \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
(X, \Lambda_X) & \xrightarrow[\;p\;]{} & (Y, \Lambda_Y)
\end{array}
$$

Assume:

- *p and q are proper (and consequently so are p_1 and p_2).*
- *p_2 is quasi-smooth.*

Then:

(i) *We have adjunctions*

$$(\mathfrak{p}_* = R\Gamma_{\Lambda_Y} \circ p_*, \mathfrak{p}^! = R\Gamma_X \circ p^!) \qquad (\mathfrak{q}_* = R\Gamma_{\Lambda_Y} \circ q_*, \mathfrak{q}^! = R\Gamma_{\Lambda'_X} \circ q^!)$$

$$(\mathfrak{p}_{1*} = R\Gamma_{\Lambda_X} \circ p_{1*}, \mathfrak{p}_1^! = R\Gamma_{\Lambda_Z} \circ p_1^!) \qquad (\mathfrak{p}_{2*} = R\Gamma_{\Lambda_{X'}} \circ p_{2*}, \mathfrak{p}_2^! = R\Gamma_{\Lambda_Z} \circ p_2^!)$$

(ii) *The diagram of pushforwards*

$$
\begin{array}{ccc}
\mathrm{QC}^!_{\Lambda_Z}(Z) & \xrightarrow{\;\mathfrak{p}_{2*}\;} & \mathrm{QC}^!_{\Lambda_{X'}}(X') \\
{\scriptstyle \mathfrak{p}_{1*}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{q}_*} \\
\mathrm{QC}^!_{\Lambda_X}(X) & \xrightarrow[\;\mathfrak{p}_*\;]{} & \mathrm{QC}^!_{\Lambda_Y}(Y)
\end{array}
$$

admits a natural equivalence

$$\mathfrak{p}_* \circ \mathfrak{p}_{1*} \simeq \mathfrak{p}_{2*} \circ \mathfrak{q}_*$$

(iii) *The diagram of pullbacks*

$$
\begin{array}{ccc}
\mathrm{QC}^!_{\Lambda_Y}(Y) & \xrightarrow{\;\mathfrak{p}^!\;} & \mathrm{QC}^!_{\Lambda_X}(X) \\
{\scriptstyle \mathfrak{q}^!}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{p}_1^!} \\
\mathrm{QC}^!_{\Lambda_{X'}}(X') & \xrightarrow[\;\mathfrak{p}_2^!\;]{} & \mathrm{QC}^!_{\Lambda_Z}(Z)
\end{array}
$$

admits a natural equivalence

$$\mathfrak{p}_1^! \circ \mathfrak{p}^! \simeq \mathfrak{q}^! \circ \mathfrak{p}_2^!$$

and is left adjointable: the resulting base-change morphism

$$\mathfrak{p}_{2*} \circ \mathfrak{p}_1^! \xrightarrow{\;\sim\;} \mathfrak{q}^! \circ \mathfrak{p}_*$$

is an equivalence.

Proof. Points (i) and (ii) are immediate from Remark 2.3.3. The functoriality equivalence of point (iii) then results from taking right adjoints.

The adjointability equivalence of point (iii) is the composite

$$\mathfrak{p}_{2*} \circ \mathfrak{p}_1^! \simeq p_{2*} \circ R\Gamma_{\Lambda_Z} \circ p_1^! \xrightarrow{\;\sim\;} R\Gamma_{\Lambda_{X'}} \circ p_{2*} \circ p_1^! \xrightarrow{\;\sim\;} R\Gamma_{\Lambda_{X'}} \circ q^! \circ p_* \simeq \mathfrak{q}^! \circ \mathfrak{p}_*$$

The second arrow is an equivalence by base-change for $\mathrm{QC}^!$; the first arrow is an equivalence by applying Lemma 2.3.1 to p_2, which is quasi-smooth and strict along each conical closed subset contained in $p_1^! \Lambda_X$, analogous to the argument in Prop. 2.3.7. $\qquad\square$

2.4. **Descent with support.** Let Δ denote the simplex category of non-empty totally ordered finite sets $[n] = \{0 \to 1 \to \cdots \to n\}$, and Δ_+ the augmented simplex category of (possibly empty) totally ordered finite sets, so in other words Δ adjoined the initial object given by the empty set $[-1] = \emptyset$.

Recall that a simplicial object or diagram of a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$, traditionally denoted by $X_\bullet$, where we understand $X_n \in \mathcal{C}$ to be the value of the functor on $[n]$. An augmented simplicial object is a functor $\Delta_+^{op} \to \mathcal{C}$, traditionally denoted by $X_\bullet \to X_{-1}$, where we understand $X_{-1} \in \mathcal{C}$ to be the value of the functor on $[-1]$.

Recall that in Δ_+ the injections $[n] \to [n+1]$ (resp. surjections $[n+1] \to [n]$), and the induced maps $X_{n+1} \to X_n$ (resp. $X_n \to X_{n+1}$) of an augmented simplicial object, are called the face (resp. degeneracy) maps. In particular, we have the distinguished face map $d_0 : X_{n+1} \to X_n$ induced by the injection $[n] \to [n+1]$ whose image does not contain $0 \in [n+1]$.

Theorem 2.4.1. *Suppose $f : (X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1})$ is an augmented simplicial diagram of maps of pairs with all stacks quasi-smooth and maps proper. Suppose further that:*

(i) The face maps are quasi-smooth.

(ii) For any map $g : [m] \to [n]$ in Δ_+, the induced commutative square

$$
\begin{array}{ccc}
(X_{n+1}, \Lambda_{n+1}) & \xrightarrow{d_0} & (X_n, \Lambda_n) \\
\tilde{g} \downarrow & & \downarrow g \\
(X_{m+1}, \Lambda_{m+1}) & \xrightarrow{d_0} & (X_m, \Lambda_m)
\end{array}
$$

is a strict Cartesian diagram of pairs.

(iii) Pullback along the augmentation

$$
\mathfrak{f}^! : \mathrm{QC}^!_{\Lambda_{-1}}(X_{-1}) \longrightarrow \mathrm{QC}^!_{\Lambda_0}(X_0)
$$

is conservative.

Then the augmentation provides an equivalence with the totalization of the cosimplicial category furnished by $!$-pullbacks with support conditions

$$
\mathrm{QC}^!_{\Lambda_{-1}}(X_{-1}) \xrightarrow{\sim} \mathrm{Tot}\{\mathrm{QC}^!_{\Lambda_\bullet}(X_\bullet), \mathfrak{f}^!_\bullet\}
$$

Proof. The first equivalence for the totalization is an application of the Beck-Chevalley Condition [Lur, Corollary 6.2.4.3] applied to the augmented cosimplicial category

$$
\left\{ \mathrm{QC}^!_{\Lambda_\bullet}(X_\bullet), \mathfrak{f}^!_\bullet \right\}
$$

The left adjointability required therein is precisely obtained by applying Prop. 2.3.8 to the diagram appearing in condition (ii) of the theorem. By hypothesis, the maps of the diagram are all proper maps of pairs, d_0 is quasi-smooth since it is a face map, and the required strictness condition holds. Thus by Prop. 2.3.8, we have the left adjointability of the diagram

$$
\begin{array}{ccc}
\mathrm{QC}^!_{\Lambda_m}(X_m) & \xrightarrow{\partial_0^!} & \mathrm{QC}^!_{\Lambda_{m+1}}(X_{m+1}) \\
\mathfrak{g}^! \downarrow & & \downarrow \tilde{\mathfrak{g}}^! \\
\mathrm{QC}^!_{\Lambda_n}(X_n) & \xrightarrow{\partial_0^!} & \mathrm{QC}^!_{\Lambda_{n+1}}(X_{n+1})
\end{array}
$$

$\square$

Corollary 2.4.2. *With the assumptions of Theorem 2.4.1, suppose furthermore that each $\mathrm{QC}^!_{\Lambda_k}(X_k)$ is compactly generated for each $k \geq 0$. (This holds, for instance, if each X_k is a global complete intersection in the sense of [AG, Section 8.2].) Then, the same holds for $k = -1$ and pushforward along the augmentation provides an equivalence*

$$
\mathrm{DCoh}_{\Lambda_{-1}}(X_{-1}) \xleftarrow{\sim} |\mathrm{DCoh}_{\Lambda_\bullet}(X_\bullet), \mathfrak{f}_{\bullet*}|
$$

Proof. By the previous theorem and the anti-equivalence of Pr^L and Pr^R, the augmented simplicial diagram

$$\mathrm{QC}^!_{\Lambda_{-1}}(X_{-1}) \longleftarrow \left\{ \mathrm{QC}^!_{\Lambda_\bullet}(X_\bullet), \mathfrak{f}_{\bullet*} \right\}$$

is a geometric realization diagram in Pr^L. The argument of [AG, Corr. 8.2.8.] identifies $\mathrm{DCoh}_{\Lambda_k}(X_k)$ with the compact objects of $\mathrm{QC}^!_{\Lambda_k}(X_k)$. Hence, since the structure maps are proper and maps of pairs, the corresponding pushforwards preserve compact objects. The result now follows from the fact the colimit of small categories exists, and the formation of Ind preserves colimits and is conservative. $\square$

In our further developments and applications in subsequent sections, we will appeal to Theorem 2.4.1 and verify its hypotheses directly. Before continuing on, let us record the following simple consequence.

Corollary 2.4.3. *Suppose S is quasi-smooth, and $\pi \colon X \to S$ is proper and quasi-smooth. Then there is a natural equivalence*

$$\mathrm{QC}(S) \simeq \mathrm{Tot}\left\{ \mathrm{QC}(X^{\times_S \bullet + 1}), f_\bullet^* \right\}$$

Proof. For each $\bullet \geq -1$, we have the identification

$$i_{\{0\}_{X^{\times_S \bullet+1}}}^! \colon \mathrm{QC}(X^{\times_S \bullet+1}) \xrightarrow{\ \sim\ } \mathrm{QC}^!_{\{0\}_{X^{\times_S \bullet+1}}}(X^{\times_S \bullet+1})$$

The left hand side furnishes the terms of a natural cosimplicial diagram with functors $f_\bullet^*$; the right hand side furnishes the terms of a natural cosimplicial diagram with functors $\mathfrak{f}_\bullet^!$. One readily checks that the latter satisfies the requirements of the preceding theorem. Thus it only remains to note that the two diagrams are intertwined by the alternative equivalences

$$\omega_{X^{\times_S \bullet+1}} \otimes - \colon \mathrm{QC}(X^{\times_S \bullet+1}) \xrightarrow{\ \sim\ } \mathrm{QC}^!_{\{0\}_{X^{\times_S \bullet+1}}}(X^{\times_S \bullet+1}) \qquad \square$$

3. Centers and traces of convolution categories

We will calculate the center and trace categories of functor categories with the composition monoidal tructure or equivalently integral kernel categories with the convolution monoidal structure.

3.1. **Preliminaries.**

Definition 3.1.1. Let $\mathcal{A}$ be an algebra object in a symmetric monoidal ∞-category $\mathcal{C}$.
   (1) The center (or Hochschild cohomology) is the morphism of bimodules object

$$Z(\mathcal{A}) = \mathrm{Hom}_{\mathcal{A}^{op} \otimes \mathcal{A}}(\mathcal{A}, \mathcal{A}) \in \mathcal{C}$$

It comes with a natural E_2-monoidal structure and universal central map $Z(\mathcal{A}) \to \mathcal{A}$.
   (2) The trace (or Hochschild homology) is the tensor of bimodules object

$$Tr(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}^{op} \otimes \mathcal{A}} \mathcal{A} \in \mathcal{C}$$

It comes with a natural S^1-action and universal trace map $\mathcal{A} \to Tr(\mathcal{A})$.

Remark 3.1.2. We refer the reader to [Lur, 6.1, 5.3] for the E_2-structure on the center (Deligne conjecture) and S^1-action on the trace (cyclic structure).

3.2. **Convolution categories.** Let $p \colon X \to Y$ be a map of derived stacks, with Y perfect and p a relative quasi-compact separated algebraic space, so that [BZNP, Theorem 3.0.4] provides an equivalence

$$\Phi \colon \mathrm{DCoh}_{p_2 - prop}(X \times_Y X) \xrightarrow{\ \sim\ } \mathrm{Fun}^{ex}_{\mathrm{Perf}\,Y}(\mathrm{Perf}\,X, \mathrm{DCoh}\,X)$$

Here we write $p_2 - prop$ instead of $prop/X$ to distinguish the second factor so there is no ambiguity.

Assume in addition X is smooth. Then on the one hand, $\mathrm{DCoh}\,X \simeq \mathrm{Perf}\,X$, so that the functor category of the right hand side

$$\mathrm{Fun}^{ex}_{\mathrm{Perf}\,Y}(\mathrm{Perf}\,X, \mathrm{DCoh}\,X) \simeq \mathrm{Fun}^{ex}_{\mathrm{Perf}\,Y}(\mathrm{Perf}\,X, \mathrm{Perf}\,X)$$

has a natural monoidal structure given by composition of linear functors, along with a natural module $\mathrm{Perf}\, X$. On the other hand, the diagonal $\Delta : X \to X \times X$ has finite tor-dimension, so that convolution equips the left hand side $\mathrm{DCoh}_{p_2-prop}(X \times_Y X)$ with a natural monoidal structure

$$(X \times_Y X) \times (X \times_Y X) \xleftarrow{\delta_{23}} X \times_Y X \times_Y X \xrightarrow{\pi_{13}} X \times_Y X \qquad \mathscr{F}_1 * \mathscr{F}_2 = \pi_{13*}\delta_{23}^*(\mathscr{F}_1 \boxtimes \mathscr{F}_2)$$

Moreover, convolution equips $\mathrm{Perf}\, X$ with a natural $\mathrm{DCoh}_{p_2-prop}(X \times_Y X)$-module structure

$$X \times (X \times_Y X) \xleftarrow{\delta_{12}} X \times_Y X \xrightarrow{\pi_3} X \qquad \mathcal{M} * \mathscr{F} = \pi_{3*}\delta_{12}^*(\mathcal{M} \boxtimes \mathscr{F})$$

(Note that p_2-properness ensures the convolution of coherent complexes and action on coherent complexes is well-defined.)

Proposition 3.2.1. *Assume X is smooth.*
Then the above equivalence is naturally a monoidal equivalence

$$\Phi : \mathrm{DCoh}_{p_2-prop}(X \times_Y X) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}\, Y}^{ex}(\mathrm{Perf}\, X, \mathrm{Perf}\, X)$$

compatibly with actions on the module $\mathrm{Perf}\, X$.

Proof. Standard base-change identities enhance the equivalence

$$\Phi : \mathrm{QC}(X \times_Y X) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{QC}(Y)}^{L}(\mathrm{QC}(X), \mathrm{QC}(X))$$

to a monoidal equivalence compatible with the actions on the module $\mathrm{QC}(X)$. The asserted monoidal equivalence is simply the restriction to full subcategories. $\qquad \square$

Corollary 3.2.2. *Assume $p : X \to Y$ is proper and X is smooth.*
Then the above equivalence is naturally a monoidal equivalence

$$\Phi : \mathrm{DCoh}(X \times_Y X) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}\, Y}^{ex}(\mathrm{Perf}\, X, \mathrm{Perf}\, X)$$

compatibly with actions on the module $\mathrm{Perf}\, X$.

3.3. **Traces of convolution categories.** Let us return to the setting of Section 2.
Assume now that X, Y are smooth and $p \colon X \to Y$ is proper (and automatically quasi-smooth).
Let $LY = \mathrm{Map}(S^1, Y)$ denote the loop space of Y. Recall the fundamental correspondence

$$X \times_Y X \xleftarrow{\delta} (X \times_Y X) \times_{X \times X} X \simeq LY \times_Y X \xrightarrow{p} LY$$

Define the support condition $\Lambda_{X/Y} \in \mathrm{Con}\, LY$ to be the the pull-push of support conditions

$$\Lambda_{X/Y} = p_* \delta^! T_{X \times_Y X}^{*-1}$$

Theorem 3.3.1. *Let X, Y be smooth and $p \colon X \to Y$ proper, quasi-smooth, and surjective.*
There is a natural cyclic identification of the trace

$$Tr(\mathrm{DCoh}(X \times_Y X)) \xrightarrow{\sim} \mathrm{DCoh}_{\Lambda_{X/Y}}(LY)$$

Proof. For notational convenience, set $\mathcal{A} = \mathrm{DCoh}(X \times_Y X)$ and $\mathcal{B} = \mathrm{Perf}\, X$. Observe that pushforward along the relative diagonal $\Delta_* \colon \mathcal{B} \to \mathcal{A}$ is monoidal, and thus we may regard $\mathcal{A}$ as an algebra in $\mathcal{B}$-bimodules.
Given an algebra $\mathcal{A}$ in $\mathcal{B}$-bimodules, we have its relative bar resolution

$$\mathcal{A} \simeq \left| \mathcal{A}^{\otimes_{\mathcal{B}}(\bullet+2)} \right|$$

which can be used to calculate its trace

$$\mathcal{A} \otimes_{\mathcal{A} \otimes \mathcal{A}} \mathcal{A} = \left| \mathcal{A}^{\otimes_{\mathcal{B}}(\bullet+2)} \right| \otimes_{\mathcal{A} \otimes \mathcal{A}} \mathcal{A} = \left| \mathcal{A}^{\otimes_{\mathcal{B}}(\bullet+1)} \otimes_{\mathcal{B} \otimes \mathcal{B}} \mathcal{B} \right|$$

We will access the trace as the geometric realization of the simplicial object

$$\mathcal{C}_\bullet = \mathcal{A}^{\otimes_{\mathcal{B}}(\bullet+1)} \otimes_{\mathcal{B} \otimes \mathcal{B}} \mathcal{B}$$
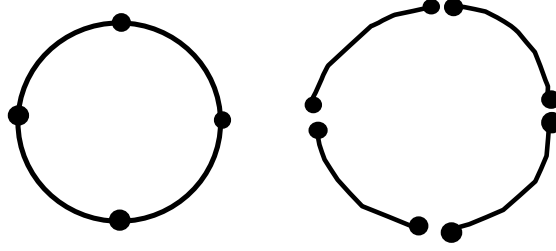
FIGURE 1. Schematic illustration of Z_n and W_n.

Unwinding the notation and using the canonical identity $\mathrm{Perf}(X)^{\otimes k} \simeq \mathrm{Perf}(X^k)$, we find the simplicial category

$$(2) \qquad\qquad . \, \mathcal{C}_\bullet = \mathrm{DCoh}(X \times_Y X)^{\otimes_{\mathrm{Perf}(X)}(\bullet+1)} \otimes_{\mathrm{Perf}(X^2)} \mathrm{Perf}\, X$$

Next we introduce the augmented simplicial diagram of derived stacks

$$Z_\bullet = X^{\times_Y \bullet+2} \times_{X^2} X \simeq X^{\times_Y \bullet+1} \times_Y LY \longrightarrow LY$$

To spell this out, identifying $[n] = \{0, \dots, n\}$ with the $(n+1)$st roots of unity in S^1, we take the relative mapping space

$$Z_n = \mathrm{Map}\left([n] \hookrightarrow S^1, X \to Y\right) = \mathrm{Map}([n], X) \times_{\mathrm{Map}([n], Y)} \mathrm{Map}(S^1, Y)$$

The simplicial structure maps come from the cosimplicial structure of the sources $[n] \hookrightarrow S^1$. Colloquially speaking, a point of Z_n is a necklace of $n+1$ points of X whose images in Y are connected by a cycle of paths; the simplicial structure maps come from forgetting or repeating points.

There is an evident fully faithful map of simplicial diagrams

$$\mathcal{C}_\bullet \longrightarrow \mathrm{DCoh}(Z_\bullet)$$

where the simplicial structure maps of the latter are pushforwards. To identify the essential image, introduce the natural level-wise maps

$$q_n : Z_n = X^{\times_Y (n+2)} \times_{X^2} X \simeq (X \times_Y X)^{\times_X (n+1)} \times_{X^2} X \longrightarrow W_n = (X \times_Y X)^{\times (n+1)}$$

obtained by taking relative diagonals, or colloquially speaking, breaking apart necklaces. Then by repeated application of Prop. 2.1.9, we have an identification of the essential image

$$\mathcal{C}_n \simeq \mathrm{DCoh}_{\Lambda_n}(Z_n) \qquad \Lambda_n = q_n^!(T^{*-1}W_n)$$

Thus we obtain an identification of simplicial diagrams

$$\mathcal{C}_\bullet \xrightarrow{\sim} \mathrm{DCoh}_{\Lambda_\bullet}(Z_\bullet)$$

Now it remains to verify the hypotheses of Theorem 2.4.1 and Cor. 2.4.2 are satisfied for the augmented simplicial diagram

$$(Z_\bullet, \Lambda_\bullet) \longrightarrow (LY, \Lambda_{X/Y})$$

(1) Proper simplicial maps, quasi-smooth face maps, and requisite Cartesian squares. Leaving aside support conditions for the moment, the augmented simplicial diagram $Z_\bullet \to LY$ is nothing more than the Cech nerve of the map $Z_0 = X \times_Y LY \to LY$, which in turn is a base change of the map $p \colon X \to Y$. Thus the face maps are proper and quasi-smooth (since p is proper and quasi-smooth), the degeneracy maps are also proper (since p is representable and separated), and the requisite squares are Cartesian (since $Z_\bullet \to LY$ is a Cech nerve).

(2) Strictness condition. Prop. 3.3.8 below verifies that the strictness condition is satisfied.

(3) Conservativity. Let $p \colon Z_0 = X \times_Y LY \to LY$ be the augmentation. Note that p is a representable proper map, so that applying [AG, Prop. 7.4.19], we are reduced to verifying that

$$\Lambda_{X/Y} = p_* \Lambda_0 = p_*(q_0)^! T^{*-1} W_0$$

20

But this is precisely the definition of $\Lambda_{X/Y}$.

(4) Compact-generation. We must also verify that $\mathrm{QC}^!_{\Lambda_n}(Z_n)$ is compactly-generated for each $n \geq 0$. Imitating the above argument, we see that the essential image of

$$\mathrm{QC}^!(X \times_Y X)^{\otimes_{\mathrm{QC}\,X}(n+1)} \otimes_{\mathrm{QC}(X)^{\otimes 2}} \mathrm{QC}\,X \longrightarrow \mathrm{QC}^!(Z_n)$$

is precisely $\mathrm{QC}^!_{\Lambda_n}(Z_n)$. Thus it is enough to observe that each of $\mathrm{QC}(X) \simeq \mathrm{QC}^!(X)$ and $\mathrm{QC}^!(X \times_Y X)$ are compactly-generated (recall our standing assumptions), and that all the monoidal/module structure maps preserve compact objects so that the various tensor products are also compactly-generated. This latter assertion follows from the smoothness and properness assumptions that we have, as was already implicit in the formula Equation 2.

This concludes the proof of Theorem 3.3.1 with the proof of Prop. 3.3.8 to appear below. $\qquad\square$

3.3.1. *Analysis of support conditions.* The goal of this and the next subsection is to establish Prop. 3.3.8. We continue with the notation introduced in the proof of Theorem 3.3.1. In this subsection, we record useful identifications of the support conditions $\Lambda_n \subset T^{*-1}Z_n$, in particular, their geometric fibers over Z_n.

First, we record the following evident descriptions of the geometric points of Z_n and W_n. Note that while $Y(k)$ is a space, the fiber product $(X \times_Y y)(k)$, for $y \in Y(k)$, is in fact a set.

Lemma 3.3.2. *Each geometric point $\eta \colon \mathrm{Spec}\,k \to Z_n$ may be represented (not necessarily uniquely) as a tuple $(y; x_0, \ldots, x_n; \ell)$ where*

$$y \in Y(k) \qquad x_0, \ldots, x_n \in (X \times_Y \{y\})(k) \qquad \ell \in \mathrm{Aut}_{Y(k)}(y)$$

Lemma 3.3.3. *Each geometric point $\eta \colon \mathrm{Spec}\,k \to W_n$ may be represented (not necessarily uniquely) as a tuple $(y_0, \ldots, y_n; x_0, x_0'; \ldots; x_n, x_n')$ where*

$$y_i \in Y(k) \qquad x_i, x_i' \in (X \times_Y \{y_i\})(k)$$

In terms of such representatives, the map $q_n : Z_n \to W_n$ is given by

$$q_n(y; x_0, \ldots, x_n; \ell) = (y, \ldots, y; x_0, x_1; x_1, x_2; \ldots; x_n, \ell \circ x_0)$$

where $\ell \circ x_0 \in (X \times_Y \{y\})(k)$ denotes the pair $(x_0, y) \in (X \times \{y\})(k)$ but where the given identification $p(x_0) \sim y \in Y(k)$ is twisted by the automorphism $\ell \in \mathrm{Aut}_{Y(k)(y)}$.

Next, we have the following description of the geometric fibers of $T^{*-1}Z_n \to Z_n$.

Lemma 3.3.4. *Fix a geometric point $\eta \colon \mathrm{Spec}\,k \to Z_n$ and a representative $(y; x_0, \ldots, x_n; \ell)$ of it. Then we have an identification of the fiber*

$$\left.(T^{*-1}Z_n)\right|_\eta = \left\{ (v_0, \ldots, v_n) \in (\Omega_Y|_y)^{\oplus(n+1)} : \begin{array}{c} (dp^*)_{x_0}(d\ell^*(v_n)) = (dp^*)_{x_0}(v_0) \\ (dp^*)_{x_1}(v_0) = (dp^*)_{x_1}(v_1) \\ \cdots \\ (dp^*)_{x_n}(v_{n-1}) = (dp^*)_{x_n}(v_n) \end{array} \right\}$$

Proof. Let us return to the necklace description of Z_n where we place X at each vertex and Y along each edge so that Z_n is the limit of the resulting finite diagram. Formation of cotangent complexes takes finite limits to finite colimits. Thus the fiber $\mathbb{L}_{Z_n}$ at a point is the colimit of the diagram where we place the appropriate fiber of $\mathbb{L}_X$ at each vertex and that of $\mathbb{L}_Y$ along each edge. Since X and Y are assumed to be

smooth, we find that the fiber is the colimit of the diagram

$$
\begin{array}{ccccccc}
& \Omega_Y|_y & & \Omega_Y|_y & & \cdots & \\
{}_{-dp^*}\swarrow & & \searrow_{dp^*} \; {}_{-dp^*}\swarrow & & \searrow_{dp^*} \; {}_{-dp^*}\swarrow & & \searrow_{dp^*} \\
\Omega_X|_{x_0} & & \Omega_X|_{x_1} & & \Omega_X|_{x_1} & & \Omega_X|_{x_n} \\
& {}^{dp^* \circ d\ell^*}\searrow & & & & \nearrow_{-dp^*} & \\
& & & \Omega_Y|_y & & &
\end{array}
$$

Taking homology gives the asserted description. $\qquad\square$

Next, we record a similar though elementary description for the geometric fibers of $T^{*-1}W_n \to W_n$.

Lemma 3.3.5. *Fix a geometric point* $\eta \colon \operatorname{Spec} k \to W_n$ *and a representative* $(y_0, \ldots, y_n; x_0, x_0'; \ldots; x_n, x_n')$ *of it. Then we have an identification of the fiber*

$$
\left. (T^{*-1}W_n) \right|_\eta = \left\{ (v_0, \ldots, v_n) \in \left(\Omega_Y|_y \right)^{\oplus(n+1)} : \begin{array}{c} (dp^*)_{x_0}(v_0) = (dp^*)_{x_0'}(v_0) = 0 \\ \cdots \\ (dp^*)_{x_n}(v_n) = (dp^*)_{x_n'}(v_n) = 0 \end{array} \right\}
$$

Finally, we arrive at the following description of the geometric fibers of $\Lambda_n \to Z_n$.

Lemma 3.3.6. *Fix a geometric point* $\eta \colon \operatorname{Spec} k \to Z_n$ *and a representative* $(y; x_0, \ldots, x_n; \ell)$ *of it.*
In terms of our previous identification of $(T^{*-1}Z_n)_\eta$, *we have an identification of the fiber*

$$
\left. (\Lambda_n) \right|_\eta = \left\{ (v_0, \ldots, v_n) \in \Omega_Y|_y^{\oplus(n+1)} : \begin{array}{c} (dp^*)_{x_0}(d\ell^*(v_n)) = (dp^*)_{x_0}(v_0) = 0 \\ (dp^*)_{x_1}(v_0) = (dp^*)_{x_1}(v_1) = 0 \\ \cdots \\ (dp^*)_{x_n}(v_{n-1}) = (dp^*)_{x_n}(v_n) = 0 \end{array} \right\} \subset \left. (T^{*-1}Z_n) \right|_\eta
$$

Proof. Let $\eta' = q_n(\eta)$. Under our previous identifications, the pullback map

$$
dq_n^* \colon Z_n \times_{W_n} \times T^{*-1}W_n \longrightarrow T^{*-1}Z_n
$$

restricted to the fibers

$$
dq_n^* \colon \left. (T^{*-1}W_n) \right|_{\eta'} \longrightarrow \left. (T^{*-1}Z_n) \right|_\eta
$$

is given by the identity

$$
dq_n^*(v_0, \ldots, v_n) = (v_0, \ldots, v_n)
$$

Thus the assertion follows from our previous identifications. $\qquad\square$

Remark 3.3.7. The previous lemmas state that

$$
\left. (\Lambda_n) \right|_\eta \subset \left. (T^{*-1}Z_n) \right|_\eta
$$

is cut out by the additional equations

$$
(dp^*)_{x_i}(v_i) = 0, \qquad \text{for all } i = 0, \ldots, n.
$$

3.3.2. *Verification of strictness condition.* We continue with the notation introduced in the proof of Theorem 3.3.1. Our goal is to complete the proof of the theorem by establishing the following.

Proposition 3.3.8. *The diagram*

$$
\begin{array}{ccc}
(Z_{n+1}, \Lambda_{n+1}) & \xrightarrow{\; d_0 \;} & (Z_n, \Lambda_n) \\
{\scriptstyle \tilde{g}} \downarrow & & \downarrow {\scriptstyle g} \\
(Z_{m+1}, \Lambda_{m+1}) & \xrightarrow{\; d_0 \;} & (Z_m, \Lambda_m)
\end{array}
$$

is a strict Cartesian diagram of pairs.

We will prove the proposition in two steps: first for face maps in Lemma 3.3.9 and then in general.

Lemma 3.3.9. *Prop. 3.3.8 holds when g corresponds to a face map (order-preserving inclusion).*

Proof. We will prove that the inclusion holds over each geometric point $\eta \in Z_{n+1}(k)$ via explicit formulas for the relevant subspaces of $(T^{*-1}Z_{n+1})\big|_\eta$. To do this, given any injection $\psi\colon [m] \to [n]$, we will give a fiberwise description of the induced map

$$Z_n \times_{Z_m} T^{*-1}Z_m \longrightarrow T^{*-1}Z_n$$

We will provide explicit formulas below, but first let us give a more informal description.

Informal description: Suppose $\eta \in Z_n(k)$ is represented by a tuple $(s; x_0, \ldots, x_n; \ell)$ as in Lemma 3.3.2. Then its image $\bar\eta \in Z_m(k)$ is given by forgetting some x_i as prescribed by ψ. The map of fibers

$$(T^{*-1}Z_m)\big|_{\bar\eta} \longrightarrow (T^{*-1}Z_n)\big|_\eta$$

is given in terms of a tuple $(v_0, \ldots, v_m)$ as in Lemma 3.3.4 by repeating entries (as in the formula for $(\tilde g)!$ below) with $d\ell^*$ inserted when looping around (as in the formula for $(d_0)^!$ below). The resulting element may be depicted graphically as follows:

Here the arrows represent summands of the linear condition cutting out $(T^{*-1}Z_m)\big|_\eta$.

Formulas: Using the above description, at a geometric point $\eta \in Z_{n+1}(k)$, we find that

$$\left((d_0)^!\Lambda_n\right)\big|_\eta = \left\{ (v_0, \cdots, v_{n+1}) \in (\Omega_Y|_y)^{\oplus(n+1)} : \begin{array}{c} v_0 = d\ell^*(v_{n+1}) \\ (d\pi^*)_{x_i}(v_{i-1}) = (dp^*)_{x_i}(v_i) = 0, \text{ for } 1 \le i \le n+1 \end{array} \right\}$$

For $\psi\colon [m] \to [n]$ the face map inducing g, the simplicial map $\tilde\psi\colon [m+1] \to [n+1]$ inducing $\tilde g$ is given by $\tilde\psi(0) = 0$ and $\tilde\psi(i) = 1 + \phi(i-1)$, for $i \ge 1$. Let $\operatorname{im}\tilde\psi \subset [n+1]$ denote the image of $\tilde\psi$, and define $a\colon [n+1] \to \operatorname{im}\tilde\psi$ by setting $a(i) = \sup\{\tilde\psi(j) : \tilde\psi(j) \le i\}$. Then we find that

$$\left((\tilde g)^!\Lambda_{m+1}\right)\big|_\eta = \left\{ (v_0, \cdots, v_{n+1}) \in (\Omega_Y|_y)^{\oplus(n+1)} : \begin{array}{c} v_i = v_{a(i)}, \text{ for } i \in [n+1] \setminus \operatorname{im}\tilde\psi \\ (dp^*)_{x_i}(v_{i-1}) = (d\pi^*)_{x_i}(v_i) = 0, \text{ for } i \in \operatorname{im}\tilde\psi \end{array} \right\}$$

Since $0 \in \operatorname{im}\tilde\psi$, we conclude that

$$((d_0)^!\Lambda_n)_\eta \cap ((p')^!\Lambda_{m+1})_\eta \subset \left\{ (v_0, \ldots, v_{n+1}) \in (T^{*-1}Z_{n+1})\big|_\eta : (dp^*)_{x_i}(v_i) = 0, \text{ for } 0 \le i \le n+1 \right\} = (\Lambda_{n+1})\big|_\eta$$

$\square$

We are now ready to complete the proof of Prop. 3.3.8 for an arbitrary simplicial structure map.

Proof of Prop. 3.3.8. Let $\psi\colon [m] \to [n]$ be the simplicial map inducing g. It uniquely factors

$$\psi\colon [m] \xrightarrow{\ \pi\ } [k] \simeq \operatorname{im}\psi \xhookrightarrow{\ \iota\ } [n]$$

as a surjection followed by an injection This gives rise to an extended diagram

$$
\begin{array}{ccc}
(Z_{n+1}, \Lambda_{n+1}) & \xrightarrow{\ d_0\ } & (Z_n, \Lambda_n) \\
\downarrow{\scriptstyle \tilde p} & & \downarrow{\scriptstyle p} \\
(Z_{k+1}, \Lambda_{k+1}) & \xrightarrow{\ d_0\ } & (Z_k, \Lambda_k) \\
\downarrow{\scriptstyle \tilde q} & & \downarrow{\scriptstyle q} \\
(Z_{m+1}, \Lambda_{m+1}) & \xrightarrow{\ d_0\ } & (Z_m, \Lambda_m)
\end{array}
$$

where p correspond to the injection ι, and q corresponds to the surjection π.

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We need to show that the large square satisfies the required strictness. By the previous lemma, we know that the top square satisfies the required strictness. Thus it suffices to show that $(\tilde{q})^!\Lambda_{m+1} = \Lambda_{k+1}$ since then

$$(\tilde{q} \circ \tilde{p})^!\Lambda_{m+1} = (\tilde{p})^!(\tilde{q})^!\Lambda_{m+1} = (\tilde{p})^!\Lambda_{k+1}$$

and we are reduced to applying the previous lemma to the top square.

To do this, as in the proof of the previous lemma, over each geometric point $\eta \in Z_{k+1}(k)$, we will give a description of the induced map of fibers

$$\left(T^{*-1}Z_{m+1}\right)\big|_\eta \longrightarrow (T^{*-1}Z_{k+1})\big|_\eta$$

Define $\pi' \colon [k] \to [m]$ to be the section of π given by its break points

$$\pi'(i) = \sup \pi^{-1}(i)$$

Then in terms of the identifications of Lemma 3.3.4, the pullback map admits the description

$$(v_0, \ldots, v_{m+1}) \longmapsto (v_0, v_{1+\pi'(0)}, \ldots, v_{1+\pi'(k)})$$

Note that this is no longer a closed immersion, and instead admits a section by repeating terms.

It is now elementary to see that $(\tilde{q})^!\Lambda_{m+1} = \Lambda_{k+1}$. On the one hand, the inclusion $(\tilde{q})^!\Lambda_{m+1} \subset \Lambda_{k+1}$ is evident. On the other hand, the inclusion $(\tilde{q})^!\Lambda_{m+1} \supset \Lambda_{k+1}$ follows from the fact that the noted section takes Λ_{k+1} into Λ_{m+1}.

This completes the proof of Prop. 3.3.8 and in turn that of Theorem 3.3.1. $\square$

3.4. **Centers of convolution categories.** The above states that $\mathrm{DCoh}(X \times_Y X)$ is like a "matrix" or endormophism algebra for $\mathrm{Perf}(X)$ over $\mathrm{Perf}(Y)$. In linear algebra, it is not hard to show that the center / Hochschild cohomology of $\mathrm{End}_R(M)$ will be the same as that of R so long as M "sees" all of R (e.g., if R is a retract of M). The main result of this section will be a categorified version of this in the special case of convolution categories. The proof will not be rather so abstract, but rather a will use the "functional analysis" for categories like DCoh developed in [BZNP].

To explain the situation, let us start with $p \colon X \to Y$ a surjective map of perfect stacks.

Observe that the loop space $LY = \mathrm{Map}(S^1, Y)$ comes equipped with a natural basepoint map $e \colon LY \to Y$ as well as rotational S^1-action. We fix once and for all the identification

$$LY \simeq Y \times_{Y \times Y} Y$$

so that e corresponds to the first projection (which is equivalent to the second projection, though in two different ways).

The category $\mathrm{QC}(LY)$ has a natural E_2-monoidal structure and the pushforward functor

$$e_* \colon \mathrm{QC}(LY) \longrightarrow \mathrm{QC}(Y)$$

realizes $\mathrm{QC}(LY)$ as the center of $\mathrm{QC}(Y)$. More generally, recall the fundamental correspondence

$$LY \xleftarrow{\ p\ } LY \times_Y X \simeq (X \times_Y X) \times_{X \times X} X \xrightarrow{\ \delta\ } X \times_Y X$$

The pullback-pushforward functor

$$\delta_* p^* \colon \mathrm{QC}(LY) \longrightarrow \mathrm{QC}(X \times_Y X)$$

also realizes $\mathrm{QC}(LY)$ as the center of $\mathrm{QC}(X \times_Y X)$.

For the above assertions, see [BZFN10]; here is an outline of a proof.

Let us recall some generalities seen in the proof of Theorem 3.3.1. For notational convenience, set $\mathcal{A} = \mathrm{QC}(X \times_Y X)$ and $\mathcal{B} = \mathrm{QC}(X)$. Observe that pushforward along the relative diagonal $\Delta_* \colon \mathcal{B} \to \mathcal{A}$ is monoidal, and thus we may regard $\mathcal{A}$ as an algebra in $\mathcal{B}$-bimodules.

Given an algebra $\mathcal{A}$ in $\mathcal{B}$-bimodules, we have its relative bar resolution

$$\mathcal{A} \simeq \left|\mathcal{A}^{\otimes_\mathcal{B}(\bullet+2)}\right|$$

which can be used to calculate its center

$$Z(\mathcal{A}) = \mathrm{Hom}_{\mathcal{A} \otimes \mathcal{A}^{op}}(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}_{\mathcal{A} \otimes \mathcal{A}^{op}}\left(\left|\mathcal{A}^{\otimes_\mathcal{B}(\bullet+2)}\right|, \mathcal{A}\right) = \mathrm{Tot}\left\{\mathrm{Hom}_{\mathcal{B} \otimes \mathcal{B}^{op}}(\mathcal{A}^{\otimes_\mathcal{B}\bullet}, \mathcal{A})\right\}$$

We will access the center as the totalization of the cosimplicial object

$$\mathcal{C}^\bullet = \mathrm{Hom}_{\mathcal{B} \otimes \mathcal{B}^{op}}(\mathcal{A}^{\otimes_{\mathcal{B}} \bullet}, \mathcal{A})$$

Unwinding the notation and using the canonical identity $\mathrm{QC}(X)^{\otimes k} \simeq \mathrm{QC}(X^k)$, we find the cosimplicial category

$$\mathcal{C}^\bullet \simeq \mathrm{Fun}^L_{\mathrm{QC}(X^2)}(\mathrm{QC}(X \times_Y X)^{\otimes_{\mathrm{QC}(X)} \bullet}, \mathrm{QC}(X \times_Y X))$$

Using the canonical identity of functors with integral transforms, we find further

$$\mathcal{C}^\bullet \simeq \mathrm{QC}(X^{\times_Y (\bullet+1)} \times_Y LY)$$

where the cosimplicial structure maps are given by $*$-pullback functors. Thus with any assumptions for which descent holds (see for example Cor. 2.4.3), one has the identification

$$Z(\mathcal{A}) = \mathrm{Tot}\, \mathcal{C}^\bullet \simeq \mathrm{QC}(LY)$$

Now let us return to small categories of coherent sheaves. Suppose now that X and Y are smooth and $p : X \to Y$ is proper and surjective. Observe that the E_2-monoidal structure on $\mathrm{QC}(LY)$ preserves the full subcategory

$$\mathrm{DCoh}_{prop/Y}(LY) \subset \mathrm{QC}(LY)$$

of coherent complexes with support proper over Y (or equivalently, proper over Y with respect to the second projection). It consists precisely of those complexes taken via e_* to the full subcategory $\mathrm{Perf}\, Y \subset \mathrm{QC}(Y)$.

Theorem 3.4.1. *Let X, Y be smooth and $p : X \to Y$ be proper and surjective.*
There is a natural E_2-identification

$$\mathrm{DCoh}_{prop/Y}(LY) \overset{\sim}{\longrightarrow} Z(\mathrm{DCoh}(X \times_Y X))$$

Proof. As explained above, via the relative bar resolution for the monoidal map $\Delta_* : \mathrm{QC}(X) \to \mathrm{QC}(X \times_Y X)$, we may calculate $Z(\mathrm{QC}(X \times_Y X))$ as the totalization of the cosimplicial diagram

$$\mathcal{C}^\bullet \simeq \mathrm{Fun}^L_{\mathrm{QC}(X^2)}(\mathrm{QC}(X \times_Y X)^{\otimes_{\mathrm{QC}(X)} \bullet}, \mathrm{QC}(X \times_Y X)) \simeq \mathrm{QC}(X^{\times_Y (\bullet+1)} \times_Y LY)$$

Likewise, via the relative bar resolution for the monoidal map $\Delta_* : \mathrm{Perf}\, X \to \mathrm{DCoh}(X \times_Y X)$, we may calculate $Z(\mathrm{DCoh}(X \times_Y X))$ as the totalization of the cosimplicial diagram

$$c^\bullet \simeq \mathrm{Fun}^{ex}_{\mathrm{Perf}(X^2)}(\mathrm{DCoh}(X \times_Y X)^{\otimes_{\mathrm{Perf}(X)} \bullet}, \mathrm{DCoh}(X \times_Y X))$$

By [BZNP, Section 5.3], the natural map of cosimplicial diagrams

$$c^\bullet \longrightarrow \mathcal{C}^\bullet$$

is fully faithful at each term thus we have a fully faithful inclusion

$$\mathrm{Tot}\, c^\bullet \hookrightarrow \mathrm{Tot}\, \mathcal{C}^\bullet \simeq \mathrm{QC}(LY)$$

The essential image consists of objects that land in $c^0 \subset \mathcal{C}^0$ under the coaugmentation map $\mathrm{Tot}\, \mathcal{C}^\bullet \to \mathcal{C}^0$. In other words, it consists of $\mathscr{F} \in \mathrm{QC}(LY)$ such that $\delta_* p^* \mathscr{F} \in \mathrm{QC}(X \times_Y X)$ in fact lies in $\mathrm{DCoh}(X \times_Y X)$. We must check that this is equivalent to $\mathscr{F} \in \mathrm{DCoh}_{prop/Y}(LY)$.

First, note that $\mathscr{F} \in \mathrm{DCoh}(LY)$ if and only if $p^* \mathscr{F} \in \mathrm{DCoh}(X \times_Y LY)$.

Let us write $W \subset LY$ for the support of $\mathscr{F}$ so that $p^{-1}W = X \times_Y W \subset X \times_Y LY$ is the support of $p^* \mathscr{F}$. Note that $W \to Y$ is proper if and only if $p^{-1}W \to Y$ is proper since $X \to Y$ is proper.

Since δ is affine, we have that $\delta_* p^* \mathscr{F} \in \mathrm{DCoh}(X \times_Y X)$ if and only if $p^* \mathscr{F} \in \mathrm{DCoh}(X \times_Y LY)$ and $p^{-1}W \to X \times_Y X$ is proper.

Finally, consider the diagram

$$p^{-1}W \longrightarrow X \times_Y X \longrightarrow Y$$

where the second map is proper since $X \to Y$ is proper. Thus $p^{-1}W \to X \times_Y X$ is proper if and only if $p^{-1}W \to Y$ and hence $W \to Y$ is proper.

We conclude that $\delta_* p^* \mathscr{F} \in \mathrm{DCoh}(X \times_Y X)$ if and only if $\mathscr{F} \in \mathrm{DCoh}(LY)$ with proper support over Y. $\square$

Remark 3.4.2. Since the theorem is the restriction of the parallel result for quasicoherent sheaves, the central functor

$$\mathrm{DCoh}_{prop/Y}(LY) \overset{\sim}{\longrightarrow} \mathrm{DCoh}(X \times_Y X)$$

is given by the pullback-pushforward functor $p_* \delta^*$ along the fundamental correspondence

$$LY \xleftarrow{\;p\;} LY \times_Y X \simeq (X \times_Y X) \times_{X \times X} X \xrightarrow{\;\delta\;} X \times_Y X$$

4. Application: affine Hecke category

We turn now to our motivating application for the development of the preceding theory.

4.1. **Global affine Hecke category.** Let G be a complex reductive group and $B \subset G$ a Borel subgroup. Let $q : BB \to BG$ denote the natural induction map of classifying stacks. Passing to loop spaces, we obtain the Grothendieck-Springer map of adjoint quotients

$$Lq : L(BB) \simeq B/B \simeq \widetilde{G}/G \longrightarrow G/G \simeq L(BG)$$

where $\widetilde{G}$ classifies pairs of a Borel subgroup $B' \subset G$ and a group element $g \in B'$, and Lq projects to the group element and forgets the Borel subgroup.

Now we will apply the preceding theory with $X = B/B$, $Y = G/G$, and $p = Lq$. Note that $X = B/B$ and $Y = G/G$ are smooth, and $p : B/B \to G/G$ is proper. Note as well that our starting point already involves loop spaces, though that structure plays no role with respect to our general results.

Definition 4.1.1. Let G be a complex reductive group and $B \subset G$ a Borel subgroup.

(i) The *global Steinberg stack* is the fiber product

$$St_G = B/B \times_{G/G} B/B$$

(ii) The *global affine Hecke category* is the small stable monoidal category

$$\mathcal{H}_G^{aff} = \mathrm{DCoh}(St_G)$$

Applying Cor. 3.2.2, we immediately obtain the following.

Theorem 4.1.2. *There is a natural monoidal equivalence*

$$\Phi : \mathcal{H}_G^{aff} = \mathrm{DCoh}(B/B \times_{G/G} B/B) \overset{\sim}{\longrightarrow} \mathrm{Fun}_{\mathrm{Perf}(G/G)}^{ex}(\mathrm{Perf}(B/B), \mathrm{Perf}(B/B))$$

compatible with actions on the module $\mathrm{Perf}(B/B)$.

4.2. **Local systems.** One can interpret the loop space $L(BG) \simeq G/G$ as the moduli stack of G-local systems on the circle S^1. Similarly, one can interpret the global Steinberg stack St_G as the moduli of G-local systems on the cylinder $S^1 \times I$ with B-reductions at the boundary circles $S^1 \times \partial I$.

Definition 4.2.1. The *commuting stack* $\mathrm{Loc}_G(T)$ is the moduli of G-local systems on the two-torus $T = S^1 \times S^1$, or equivalently the twice-iterated loop space

$$\mathrm{Loc}_G(T) \simeq L(L(BG))$$

Remark 4.2.2. The name commuting stack comes from the presentation

$$\mathrm{Loc}_G(T) \simeq \left\{ (g_1, g_2) \in G \times G \,|\, g_1 g_2 g_1^{-1} g_2^{-1} = 1 \right\} / G$$

One should be careful to understand that the commutator equation needs to be imposed in a derived sense.

Let $\mathfrak{g}$ denote the Lie algebra of G. The fiber of the cotangent complex of $\mathrm{Loc}_G(T)$ at a local system $\mathscr{P}$ can be calculated by the de Rham cochains $C^*(T, \mathfrak{g}_{\mathscr{P}}^*)[1]$, where $\mathfrak{g}_{\mathscr{P}}^*$ denotes the coadjoint bundle of $\mathscr{P}$. Focusing on the degree -1 term coming from the commutator equation, we see that there is a natural map

$$\mu|_{\mathscr{P}} : T_{\mathrm{Loc}_G(T)}^{*-1}\Big|_{\mathscr{P}} \simeq H^0(T, \mathfrak{g}_{\mathscr{P}}^*) \longrightarrow \mathfrak{g}^*/G$$

Let $\mathfrak{h}$ denote the Lie algebra of the universal Cartan of G, and W the Weyl group. Recall the dual characteristic polynomial map, or equivalently, the projection to the coadjoint quotient

$$\chi : \mathfrak{g}^*/G \longrightarrow \mathfrak{g}^*/\!/G \simeq \mathfrak{h}^*/W$$

Definition 4.2.3. The *global nilpotent cone* $\mathcal{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$ is the conic closed subset given by the inverse-image of zero under the composition

$$T^{*-1}_{\mathrm{Loc}_G(T)} \xrightarrow{\ \mu\ } \mathfrak{g}^*/G \xrightarrow{\ \chi\ } \mathfrak{h}^*/W$$

4.3. Center of affine Hecke category. Starting from the loop space $\mathrm{Loc}_G(S^1) \simeq L(BG) \simeq G/G$, we have arrived at the commuting stack $\mathrm{Loc}_G(T) \simeq L(L(BG))$ by taking loops again. Thus there is a natural asymmetry to the construction: we will distinguish the projection to the first loop, or in other words, the basepoint of the second loop

$$\mathrm{Loc}_G(T) \longrightarrow \mathrm{Loc}_G(S^1)$$

Following our general results, we introduce the full subcategory

$$\mathrm{DCoh}_{prop/\,\mathrm{Loc}_G(S^1)}(\mathrm{Loc}_G(T)) \subset \mathrm{DCoh}(\mathrm{Loc}_G(T))$$

of coherent complexes with proper support along the projection to the first loop.

Applying Theorem 3.4.1, we immediately obtain the following.

Theorem 4.3.1. *There is a natural E_2-monoidal identification of the center*

$$Z(\mathrm{DCoh}(B/B \times_{G/G} B/B)) \simeq \mathrm{DCoh}_{prop/\,\mathrm{Loc}_G(S^1)}(\mathrm{Loc}_G(T))$$

Remark 4.3.2. The description of the center is manifestly not $\mathrm{SL}_2(\mathbb{Z})$-equivariant in contrast to that of the trace calculated below.

4.4. Trace of affine Hecke category. Recall that we have introduced the global nilpotent cone $\mathcal{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$, and can consider the corresponding full subcategory

$$\mathrm{DCoh}_{\mathcal{N}}(\mathrm{Loc}_G(T)) \subset \mathrm{DCoh}(\mathrm{Loc}_G(T))$$

of coherent complexes supported along it. All of these constructions are manifestly $\mathrm{SL}_2(\mathbb{Z})$-equivariant.

Theorem 4.4.1. *There is a canonical S^1-equivariant identification of the trace*

$$Tr(\mathrm{DCoh}(B/B \times_{G/G} B/B)) \simeq \mathrm{DCoh}_{\mathcal{N}}(\mathrm{Loc}_G(T))$$

Proof. Set $X = L(BB) \simeq B/B$, $Y = L(BG) \simeq G/G$, and $p = Lq$ for $q : BB \to BG$.

Applying Theorem 3.3.1, it remains to identify the support condition $\Lambda_{X/Y} \subset T^{*-1}_{LY}$ as described therein with the global nilpotent locus $\mathcal{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$.

Recall $\mathfrak{g}$ denotes the Lie algebra of G. Let $\mathfrak{b}$ denote the Lie algebra of B. Let $N \subset B$ denote the maximal unipotent subgroup, and $\mathfrak{n}$ its Lie algebra. Fix an invariant inner product on $\mathfrak{g}$, so that we have an identification $\mathfrak{g} \simeq \mathfrak{g}^*$, and in particular an identification $\mathfrak{n} = \ker(\mathfrak{g}^* \to \mathfrak{b}^*)$

Fix a geometric point $\eta : \mathrm{Spec}\, k \to \mathrm{Loc}_G(T) \simeq L(L(BG))$ given by a pair of commuting elements $(\alpha, \beta) \in G \times G$. Note that the intermediate stack $LY \times_Y X$ is the moduli of G-bundles on $S^1 \times S^1$ with a B-reduction along the first loop. Thus we have the identifications:

$$T^{*-1}_{\mathrm{Loc}_G(T)}\Big|_{\eta} = \{v \in \mathfrak{g}^* \simeq \mathfrak{g}: \ \mathrm{ad}_\alpha v = \mathrm{ad}_\beta v = v\}$$

$$\Lambda_{X/Y}\big|_{\eta} = \left\{v \in T^{*-1}_{\mathrm{Loc}_G(T)}\Big|_{\eta} : \exists g \in G \text{ such that } \mathrm{ad}_g\, \alpha \in B, \mathrm{ad}_g\, v \in \mathfrak{n})\right\}$$

Recall that $v \in \mathfrak{g}^* \simeq \mathfrak{g}$ is nilpotent if and only if there exists $g \in G$ such that $\mathrm{ad}_g\, v \in \mathfrak{n}$. Thus we clearly have the containment:

$$\Lambda_{X/Y}\big|_{\eta} \subset \mathcal{N}\big|_{\eta} = \left\{v \in T^{*-1}_{\mathrm{Loc}_G(T)}\Big|_{\eta} : \exists g \in G \text{ such that } \mathrm{ad}_g\, v \in \mathfrak{n})\right\}$$

Thus it suffices to show that two commuting elements $v \in \mathfrak{g}$, $\alpha \in G$ with v nilpotent are contained in a Borel subgroup $B \subset G$. Equivalently, it suffices to show two such elements simultaneously fix a point of the flag variety G/B. Note that a nilpotent element $v \in \mathfrak{g}$ generates an $\mathbb{A}^1$-action on G/B, and the action preserves the fixed points of the element $\alpha \in G$. Since the fixed points of α are a nonempty projective variety, the $\mathbb{A}^1$-action must have a fixed point. This concludes the proof. $\qquad\square$

Remark 4.4.2. Note that the map $\mathrm{Loc}_B(T) \to \mathrm{Loc}_G(T)$ is not necessarily surjective, even though $B/B \to G/G$ always is. For example, consider $G = \mathrm{PGL}_2$ and two commuting elements $\alpha, \beta \in \mathrm{PGL}_2$ that are not contained in any Borel subgroup $B \subset G$. However, if the derived group of G is simply-connected, then the map is in fact surjective. In this case, one can derive the global nilpotent cone $\mathscr{N} \subset T^{*-1}_{\mathrm{Loc}_G(T)}$ directly from the map $\mathrm{Loc}_B(T) \to \mathrm{Loc}_G(T)$.

References

[AG]   D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves, and the geometric langlands conjecture*, available at arxiv:1201.6343.

[BD]   A. Beilinson and V. Drinfeld, *Quantization of hitchin hamiltonians and hecke eigensheaves*.

[Bez06]   Roman Bezrukavnikov, *Noncommutative counterparts of the Springer resolution*, International Congress of Mathematicians. Vol. II, 2006, pp. 1119–1144.

[Bez]   R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, available at arxiv:1209.0403.

[BFO12]   R. Bezrukavnikov, M. Finkelberg, and V. Ostrik, *Character D-modules via Drinfeld center of Harish-Chandra bimodules*, 2012, pp. 589–620.

[BIK08]   Dave Benson, Srikanth B. Iyengar, and Henning Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 4, 573–619.

[BKV]   R. Bezrukavnikov, D. Kazhdan, and Y. Varshavsky, *On the stable center conjecture*, available at arxiv:1307.4669.

[BZFN10] David Ben-Zvi, John Francis, and David Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966, available at arxiv:0805.0157.

[BZHN14] D. Ben-Zvi, D. Helm, and D. Nadler, *Deligne-Langlands in families and coherent springer theory* (2014), In preparation.

[BZNa]   D. Ben-Zvi and D. Nadler, *The character theory of a complex group*, available at arxiv:math/0904.1247.

[BZNb]   _____, *Elliptic springer theory*, available at arxiv:math/1302.7053.

[BZNc]   _____, *Loop spaces and connections*, J. Topology, available at arxiv:1002.3636.

[BZNd]   _____, *Loop spaces and representations*, Duke Math J., available at arxiv:1004.5120.

[BZNP]   D. Ben-Zvi, D. Nadler, and A. Preygel, *Integral transforms for coherent sheaves*.

[CG97]   Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[DG]   V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*.

[KL87]   David Kazhdan and George Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215.

[Lur]   Jacob Lurie, *Higher algebra*. http://math.harvard.edu/~lurie/papers/HigherAlgebra.pdf.

[Lusa]   G. Lusztig, *Unipotent almost characters of simple p-adic groups*, available at arxiv:1212.6540.

[Lusb]   _____, *Unipotent almost characters of simple p-adic groups ii*, available at arxiv:1303.5026.

[Pre11]   Anatoly Preygel, *Thom-Sebastiani & duality for matrix factorizations*, arXiv e-print (January 2011), available at arxiv:1101.5834.

[SV11]   O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math **147** (2011), no. 1, 188234.

[SV12]   _____, *Hall algebras of curves, commuting varieties and Langlands duality*, Math. Ann **353** (2012), no. 4, 1399–1451.

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