VANISHING THEOREMS AND THE HODGE $\Rightarrow$ DE RHAM SPECTRAL SEQUENCE

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1. KAN vanishing

Every algebraic geometer will recognize

Theorem 1.1 (Kodaira). *Suppose* X *a compact Kahler variety,* $\mathscr{L}$ *a positive line bundle. Then,* $H^q(X, \Omega_X^{\wedge \dim X} \otimes \mathscr{L}) = 0$ *for all* $q > 0$ *(and equivalently by Serre Duality:* $H^{q'}(X, \mathscr{L}^\vee) = 0$ *for all* $q' < \dim X$*).*

and in fact

Theorem 1.2 (Kodaira-Akizuki-Nakano). *Suppose* X *a compact Kahler variety,* $\mathscr{L}$ *a positive line bundle. Then,* $H^q(X, \Omega_X^{\wedge r} \otimes \mathscr{L}) = H^{q'}(X, \Omega_X^{\wedge r'} \otimes \mathscr{L}^\vee) = 0$ *for* $q + r > \dim X$, $q' + r' < \dim X$.

From this complex analytic statement we can deduce

$$\{\text{“}X \text{ compact Kahler, } \mathscr{L} \text{ positive”}\} \overset{\textbf{GAGA}}{\Rightarrow} \{\text{“}X \text{ smooth projective } \mathbb{C}\text{-scheme, } \mathscr{L} \text{ ample”}\}$$

$$\overset{\textbf{“Lefschetz Principle”}}{\Rightarrow} \{\text{“}X \text{ smooth projective } k\text{-scheme, } k \text{ characteristic 0 field, } \mathscr{L} \text{ ample”}\}$$

Theorem 1.3 (Algebraic Kodaira-Akizuki-Nakano). *Suppose* X *is a smooth projective* k*-scheme (*k *char. 0 field),* $\mathscr{L}$ *ample line bundle. Then,* $H^q(X, \Omega_X^{\wedge r} \otimes \mathscr{L}) = H^{q'}(X, \Omega_X^{\wedge r'} \otimes \mathscr{L}^\vee) = 0$ *for* $q + r > \dim X$, $q' + r' < \dim X$.

Let's reflect on this a bit:

- **The Good:** This is a vanishing theorem in characteristic 0, whose statement is purely algebraic;
- **The Bad:** The proof indicated above was transcendental in nature (the proof over $\mathbb{C}$ uses Hodge theory, with its elliptic operators and harmonic representatives).
- **The Ugly:** The statement can *fail* in characteristic $p > 0$: (Some Enriques surfaces in char. 2, other examples. Raynaud and others.)

Deligne-Illusie-Raynaud tell us how to get around "The Bad" (and learn to live with "The Ugly") and in fact give a remarkably elementary proof of KAN vanishing. Before discussing it further, let's define the other part of the title:

2. De Rham cohomology, Hodge-deRham spectral sequence

What do we mean by the de Rham cohomology of a scheme?

Definition 2.1. Suppose X is a smooth Y-scheme. Then, we have the *de Rham complex* $\Omega^\bullet_{X/Y}$:

$$\Omega^i_{X/Y} \overset{\text{def}}{=} \bigwedge^i \Omega_{X/Y} \qquad \text{and} \qquad d : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$$

where d is the usual "exterior derivative." Note that each $\Omega^i_{X/Y}$ is an $\mathcal{O}_X$-module, but d is only $f^{-1}\mathcal{O}_Y$-linear. The *de Rham cohomology of* X is defined to be the hypercohomology of $\Omega^\bullet_{X/Y}$.

$$H^i_{dR}(X/Y) \overset{\text{def}}{=} \mathbb{H}^i(X, \Omega^\bullet_{X/Y})$$

If you've never seen this before, this might seem strange. Here's a bit of motivation:

Example 2.2.
- Suppose Y is a manifold, $\Omega^\bullet_Y$ the *smooth, $\mathbb{C}$-valued* de Rham complex. By the (smooth) Poincaré Lemma, $\underline{\mathbb{C}} \to \Omega^\bullet_Y$ is a quasi-isomorphism (i.e., the de Rham complex is a resolution of the constant sheaf $\underline{\mathbb{C}}$). But now, $\Omega^\bullet_Y$ are modules over the *fine* sheaf of algebras $\mathcal{O}_Y^{C^\infty}$, so that they are themselves fine. Thus, $H^i(Y, \underline{\mathbb{C}}) = \mathbb{H}^i(Y, \Omega^\bullet_Y) = H^i(\Gamma(Y, \Omega^\bullet_Y))$. Someone who's only seen this example might want to define the de Rham cohomology of X/Y as just $H^i(\Omega^\bullet_{X/Y})$. Here's why that's not a good idea:
- Suppose X is a finite type smooth $\mathbb{C}$-scheme, with X^{an} the associated analytic space, and $\Omega^\bullet_{X^{an}}$ the *holomorphic* de Rham complex. By the (holomorphic) Poincaré Lemma, $\underline{\mathbb{C}} \to \Omega^\bullet_{X^{an}}$ is a quasi-isomorphism, and as before $H^i(X^{an}, \underline{\mathbb{C}}) = \mathbb{H}^i(X^{an}, \Omega^\bullet_{X^{an}})$. But, in this case the resolution *need not be acyclic* so we can't merely take $H^i \circ \Gamma$. In fact, by a theorem of Grothendieck the natural map $\mathbb{H}^i(X, \Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^i(X^{an}, \Omega^\bullet_{X^{an}}) \cong H^i(X^{an}, \underline{\mathbb{C}})$ under our hypotheses. So, $H^\bullet_{dR}(X/\mathbb{C})$ is a purely algebraic object which recovers the singular cohomology for smooth finite-type $\mathbb{C}$ schemes—our definition is definitely reasonable.

Definition 2.3. We have the *hypercohomology spectral sequence* for $\Omega^\bullet_{X/Y}$:

$$E_1^{p,q} = H^q(X, \Omega^p_{X/Y}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/Y}) = H^{p+q}_{dR}(X/Y).$$

This is called the *Hodge-de Rham spectral sequence.*

Remark 2.4. Suppose that $Y = \operatorname{Spec} k$ (k a field) and that $X \to Y = \operatorname{Spec} k$ is proper. Then, all terms on page 1 of the spectral sequence are finite-dimensional k-vector spaces. Since each successive page involves passing to subquotients, we obtain the following (well-defined) inequality

$$\sum_{p+q=n} \dim_k H^q(X, \Omega^p_{X/Y}) \geqslant \dim_k H^n_{dR}(X/Y).$$

Moreover, we see that *equality holds iff all differentials vanish*, i.e., iff the spectral sequence *degenerates at page* 1.

Example 2.5. Set $k = \mathbb{C}$, $X/\mathbb{C}$ a finite-type $\mathbb{C}$-scheme, and X^{an} the associated analytic space. The Dolbeault Lemma shows that $\Omega^\bullet_{X^{an}} \to \operatorname{Tot} \mathcal{A}^{\bullet,\bullet}$ is a (filtered) quasi-isomorphism of complexes of sheaves where the right hand complex is fine (and thus acyclic for $\Gamma(X, -)$). This yields an isomorphism of spectral sequences

$$\left\{H^q(X, \Omega^p_{X^{an}}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X^{an}})\right\} \cong \left\{H^q(\Gamma(X, \mathcal{A}^{p,\bullet})) \Rightarrow H^{p+q}(\Gamma(X, \operatorname{Tot} \mathcal{A}^{\bullet,\bullet}))\right\}$$

But, Hodge theory (in particular existence of harmonic representatives) shows that $\operatorname{Tot} \mathscr{H}^{\bullet,\bullet} \to \Gamma(X, \operatorname{Tot} \mathcal{A}^{\bullet,\bullet})$ is a (filtered) quasi-isomorphism of complexes *where the left hand complex has no non-zero differentials* (here, $\mathscr{H}^{\bullet,\bullet}$ is the bigraded space of harmonic forms). This induces an isomorphism of spectral sequences

$$\left\{H^q(\Gamma(X, \mathcal{A}^{p,\bullet})) \Rightarrow H^{p+q}(\Gamma(X, \operatorname{Tot} \mathcal{A}^{\bullet,\bullet}))\right\} \cong \left\{H^1(\mathscr{H}^{p,\bullet}) \Rightarrow H^{p+q}(\operatorname{Tot} \mathscr{H}^{\bullet,\bullet})\right\}.$$

Since all differentials in $\operatorname{Tot} \mathscr{H}^{\bullet,\bullet}$ are zero, certainly the same is true for its spectral sequence—which must then degenerate at page 1.

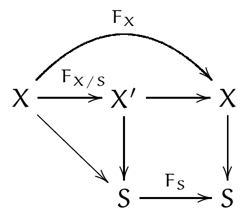
Definition 2.6. Suppose $C^\bullet$ is a complex (of some sort). We say that $C^\bullet$ is *decomposable* if it is quasi-isomorphic to a complex with no non-zero differentials. (Applying Remark 2.4 we see that the hypercohomology spectral sequence of any such complex degenerates—at least under suitable finiteness assumptions—though certainly the converse does not necessarily hold.)

3. "The Bad," contd.

Work of Esnault and Viehweg showed that KAN-type vanishing results can be deduced directly from degeneration of certain "Hodge-de Rham"-*like* spectral sequences. There were proofs of degeneration of the Hodge-de Rham spectral sequence (not necessarily the related sequences of Esnault-Viehweg) in characteristic 0 avoiding complex Hodge theory before Deligne-Illusie: c. 1984/5 via Faltings' work on the Hodge-Tate decomposition for $\mathbb{Q}_p$-étale cohomology.[1] Deligne-Illusie(-Raynaud) in fact accomplish two things (each one involving a separate bit of characteristic p goodness!):

- They give an elementary argument proving degeneration (under certain hypotheses) in characteristic p, and deduce degeneration in characteristic 0 by "standard arguments."
- They give a direct simple argument (relying on being in char. p) for a modified version of KAN vanishing in characteristic p, and deduce KAN vanishing in characteristic 0 by "standard arguments."

The first bit of characteristic p goodness allows Deligne-Illusie to prove the following statement which fairly easily implies an analog of Hodge-de Rham degeneration in characteristic p. Before stating it, let's recall some of the basics of Frobenius, and put up a diagram reminind us of where everything is: Suppose S is an $\mathbb{F}_p$ scheme, and X an S-scheme. We have absolute Frobenius maps $F_X : X \to X$ and $F_S : S \to S$. There is also a relative Frobenius map $F = F_{X/S} : X \to X'$ defined by the following commutative diagram with cartesian square



Now, we can state:

Theorem 3.1. *Let S be an $\mathbb{F}_p$-scheme, $\widetilde{S}$ a (flat) lifting of S over $\mathbb{Z}/p^2$. Suppose that X is a smooth S-scheme, and $F : X \to X'$ the relative Frobenius. Then, the following are equivalent:*

(a) *X' admits a flat lifts to(/deformation over) $\widetilde{S}$;*
(b) *$\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$ is decomposable in $D(X')$ (Yes! It's an element of $D(X')$! More on this in Section 4.);*
(c) *$\tau_{<p} F_* \Omega^\bullet_{X/S}$ is decomposable in $D(X')$;*
(d) *There is a quasi-isomorphism in $D(X')$*

$$\bigoplus_{i<p} \Omega^i_{X'/S}[-i] \cong \tau_{<p} F_* \Omega^\bullet_{X/S}.$$

Instead of proving it, I'll explain the bit of characteristic p goodness, justify the equivalence of the last two points, and give a strong argument for expecting the first two points to be equivalent (leaving the rest for Jay).

This turns out to be enough to recover limited characteristic p analogs of degeneration of the Hodge-de Rham spectral sequence and KAN vanishing. To do so, we'd apply the Theorem in the case of

Example 3.2. Take $S = \operatorname{Spec} k$, with k a perfect field of char. $p > 0$. Then, $\widetilde{S} = \operatorname{Spec} W_2(k)$ is a flat lifting of k over $\mathbb{Z}/p^2 = W_2(\mathbb{F}_p)$. (In fact, it is the unique such up to isomorphism by general deformation theory since $\operatorname{Spec} k$ is affine and S/k is smooth.[2]) In this case, $F_S : S \to S$ is an isomorphism, and lifting X/S and X'/S is equivalent.

We can deduce a characteristic p analog of KAN vanishing directly from the above theorem using an easy argument that's very dependent on the fact that we're in characteristic p:

Theorem 3.3 (Deligne-Illusie-Raynaud). *Let $S = \operatorname{Spec} k$, for k a perfect field of characteristic $p > 0$, and $\widetilde{S} = \operatorname{Spec} W_2(k)$. Suppose X is a smooth projective k-scheme that admits a lifting to $W_2(k) = \widetilde{S}$, and $\mathscr{L}$ an ample invertible sheaf on X. Then, $H^q(X, \Omega^r_X \otimes \mathscr{L}) = H^{q'}(X, \Omega^{r'}_X \otimes \mathscr{L}^\vee) = 0$ for $q + r >$ $\max\{\dim X, 2\dim X - p\}$, $q' + r' < \min\{\dim X, p\}$. (Here, $q + q' = r + r' = \dim X$.)*

[1] Still not algebraic: This approach uses p-adic analysis and the comparison theorem for étale and Betti cohomology.
[2] Space of lifts is a torsor for $H^1(\operatorname{Spec} k, T_{S/k}) = 0$.

Proof. By Serre duality it suffices to prove the second part: that $H^{q'}(X, \Omega_X^{r'} \otimes \mathscr{L}^\vee) = 0$ for $q' + r' < \min\{\dim X, p\}$. By Serre vanishing, there is some n so that $H^{q'}(X, \Omega_X^{r'} \otimes \mathscr{L}^{\otimes p^n}) = 0$ for all $q' > 0$ and all r'. By Serre duality, $H^q(X, \Omega_X^r \otimes \mathscr{L}^{\otimes -p^n}) = 0$ for all $q < \dim X$ and all r, and in particular whenever $q + r < \min\{\dim X, p\}$. (That back and forth made my head spin!)

Now, by induction on n (with $\mathscr{M} = \mathscr{L}^{\otimes -p^{n-1}}$), it suffices to prove that

Claim: *Suppose that $\mathscr{M}$ is a line bundle satisfying $H^q(X, \Omega_X^r \otimes \mathscr{M}^p) = 0$ whenever $q + r < \min\{\dim X, p\}$. Then, $H^q(X, \Omega_X^r \otimes \mathscr{M}) = 0$ whenever $q + r < \min\{\dim X, p\}$.*

Proof of Claim:

Set $\mathscr{M}' = \mathscr{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$, and consider the hypercohomology spectral sequence for $\mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^\bullet_{X/k}$:

$$H^q(X', \mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^r_{X/k}) \Rightarrow \mathbb{H}^{q+r}(X', \mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^\bullet_{X/k}).$$

Note that $F^* \mathscr{M}' = F_X^* \mathscr{M} = \mathscr{M}^{\otimes p}$, so that the projection formula yields

$$\mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^r_{X/k} = F_*(F^* \mathscr{M}' \otimes_{\mathcal{O}_X} \Omega^r_{X/k}) = F_*(M^{\otimes p} \otimes_{\mathcal{O}_X} \Omega^r_{X/k}) \qquad \text{for all } r$$

Thus, the hypotheses of the claim imply that all terms on page 1 of the spectral sequence with $q + r < \min\{\dim X, p\}$ vanish, and so

$$0 = \mathbb{H}^n(X', \mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^\bullet_{X/k}) \qquad \text{for } n < \min\{\dim X, p\}.$$

Since $\mathscr{M}'$ is a line bundle, it is flat: So, $\mathscr{M}' \otimes -$ is an exact functor on the derived category, and in particular it preserves quasi-isomorphism and behaves well with respect to truncation. By Theorem 3.1 we have a quasi-isomorphism

$$\bigoplus_{i<p} \Omega^i_{X'/k}[-i] \cong \tau_{<p} F_* \Omega^\bullet_{X/k}$$

and so for $n < \min\{\dim X, p\}$ we have an isomorphism

$$0 = \mathbb{H}^n(X', \mathscr{M}' \otimes_{\mathcal{O}_{X'}} F_* \Omega^\bullet_{X/k}) \cong \oplus_i H^{n-i}(X', \mathscr{M}' \otimes_{\mathcal{O}_{X'}} \Omega^i_{X'/k}) = \oplus_i H^{n-i}(X, \mathscr{M} \otimes_{\mathcal{O}_X} \Omega^i_{X/k}) \otimes_k k,$$

where the last equality (with the map $k \to k$ given by Frobenius) follows by base-change (since $X' = X \times_S S$, with $S \to S$ given by F_S). This proves the claim, and the theorem. □

4. Cartier isomorphism

We begin with an observation. Suppose $x \in \Gamma(U, F_* \Omega^\bullet_{X/S})$ and $a \otimes k \in \Gamma(U, \mathcal{O}_{X'})$ are local sections. Then:

$$d\left((a \otimes k) \cdot_F x\right) = d(k a^p x) = k p a^{p-1} d(x) + k a^p d(x) = k a^p d(x) = (a \otimes k) \cdot d(x).$$

So, the exterior derivative on $F_* \Omega^\bullet_{X/S}$ is $\mathcal{O}_{X'}$*-linear*, and $F_* \Omega^\bullet_{X/S}$ is a complex of $\mathcal{O}_{X'}$-modules! Then, a natural question is to ask what the cohomology sheaves of this complex are. This is answered by:

Theorem 4.1 (Cartier). *Let S be an $\mathbb{F}_p$-scheme and X an S-scheme, $F : X \to X'$ the relative Frobenius/S. Then, there is a homomorphism of graded $\mathcal{O}_{X'}$-algebras*

$$C^{-1} : \bigoplus \Omega^i_{X'/S} \to \bigoplus \mathcal{H}^i(F_* \Omega^\bullet_{X/S})$$

uniquely characterized by $C^{-1}(F^*(ds)) = s^{p-1} ds$ for every section s of $\Omega^1_{X/S}$. Moreover, this map is an isomorphism if X/S is smooth.

Remark 4.2. This obviously implies the equivalence of (c) and (d) in Theorem 3.1. Another, slightly less obvious but also useful, consequence is that $F_* \Omega^i_{X/S}$ is locally free.

Key Fact (which Jay will show next time): If $F : X \to X'$ lifts to $\widetilde{S}$. Then, one can construct a chain map

$$\bigoplus \Omega^i_{X'/S}[i] \to F_* \Omega^\bullet_{X/S}$$

inducing C^{-1} on cohomology.

5. Decomposing $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$ and liftings

(Notation: $\mathcal{A}$ an abelian category, $C(\mathcal{A})$ the category of cochain complexes, $D = D(\mathcal{A})$ the derived category.) **Question:** Suppose $C^\bullet = \{C^0 \xrightarrow{d} C^1\}$ is a two-term complex in $C(\mathcal{A})$. When is $C^\bullet$, as object of $D(\mathcal{A})$, decomposable?
Associated to $C^\bullet$ is the distinguished triangle

$$\tau_{\leqslant 0} C^\bullet \to \tau_{\leqslant 1} C^\bullet \to H^1 C^\bullet[-1] \to \tau_{\leqslant 0}[1]$$

which we may canonically identify with the distinguished triangle

$$H^0 C^\bullet \to C^\bullet \to H^1 C^\bullet[-1] \to H^0 C^\bullet[1]$$

Via the identification

$$\mathrm{Hom}_D(H^0 C^\bullet \oplus H^1 C^\bullet[-1], C^\bullet) = \mathrm{Hom}_D(H^0 C^\bullet, C^\bullet) \times \mathrm{Hom}_D(H^1 C^\bullet[-1])$$

we see that the data of a map $H^0 C^\bullet \oplus H^1 C^\bullet[-1] \to C^\bullet$ inducing the identity on cohomology is the same as an element of $\mathrm{Hom}_D(H^1 C^\bullet[-1], C^\bullet)$ mapping to $\mathrm{id} \in \mathrm{Hom}_D(H^1 C^\bullet[-1], H^1 C^\bullet[-1])$ via the map in the above distinguished triangle.

Definition 5.1. Suppose $C^\bullet$ in $C(\mathcal{A})$ is concentrated in degrees $0, 1$. A *decomposition* (or *decomposition map*) of $C^\bullet$ is an arrow $H^1 C^\bullet[-1] \to C^\bullet$ of $D(\mathcal{A})$ mapping to $\mathrm{id} \in \mathrm{Hom}_D(H^1 C^\bullet[-1], H^1 C^\bullet[-1])$ via the natural map above. (Foreshadowing definition: A morphism $H^1 C^\bullet[-1] \to C^\bullet$ of $C(\mathcal{A})$, that is a decomposition when regarded as a morphism in $D(\mathcal{A})$, is called a *strict decomposition (map).*)

The long exact sequence associated to the distinguished triangle includes the fragment

$$0 \to \mathrm{Hom}_D(H^1 C^\bullet[-1], H^0 C^\bullet) \to \mathrm{Hom}_D(H^1 C^\bullet[-1], C^\bullet) \xrightarrow{a} \mathrm{Hom}_D(H^1 C^\bullet[-1], H^1 C^\bullet[-1]) \xrightarrow{\delta} \mathrm{Hom}_D(H^1 C^\bullet[-1], H^0 C^\bullet[1]).$$

So, we see that a decomposition map exists iff $\delta(\mathrm{id}) = 0 \in \mathrm{Hom}_D(H^1 C^\bullet[-1], H^0 C^\bullet[1]) = \boxed{\mathrm{Ext}^2(H^1 C^\bullet, H^0 C^\bullet)}$.
If a decomposition exists, then the set of all decompositions ($a^{-1}(\mathrm{id})$) is a torsor for $\ker a = \mathrm{Hom}_D(H^1 C^\bullet[-1], H^0 C^\bullet) = \boxed{\mathrm{Ext}^1(H^1 C^\bullet, H^0 C^\bullet)}$.

What about $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$?
In this case, the obstruction lies in (using the Cartier isomorphism)

$$\mathrm{Ext}^2(H^1 \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}, H^0 \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}) = \mathrm{Ext}^2(H^1 F_* \Omega^\bullet_{X/S}, H^0 F_* \Omega^* \bullet_{X/S}) = \mathrm{Ext}^2(\Omega^1_{X'/S}, \mathcal{O}_{X'}) = H^2(X', T_{X'/S}),$$

and the decomposing maps form a torsor for

$$\mathrm{Ext}^1(H^1 \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}, H^0 \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}) = H^1(X', T_{X'/S}).$$

Observation: Those looks like groups we recognize from deformation theory! Which groups?
The obstruction to lifting X'/S to $\widetilde{S}$ lies in $\mathrm{Ext}^2(\Omega^1_{X'/S}, \mathcal{O}_{X'}) = H^2(X', T_{X'/S})$. These liftings are then a torsor for $\mathrm{Ext}^1(\Omega^1_{X'/S}, \mathcal{O}_{X'}) = H^1(X', T_{X'/S})$. Then, the automorphism group of each lifting is isomorphic to $\mathrm{Hom}(\Omega^1_{X'/S}, \mathcal{O}_{X'}) = H^0(X', T_{X'/S})$. This provides a first bit of evidence for (a)⇔(b) of Theorem 3.1. (In fact, (a)⇔(b) is precisely the statement that the two obstructions always either both vanish or both don't.)

6. Local is easier than global

Goal: We want to construct a bijection between the liftings and splittings. (In fact, by the Theorem combined with the above "is a torsor for" computations, they should be in bijection.) Working globally all at once is hard (even just constructing liftings is), and we have some intuition that this sort of obstruction/torsor/automorphism setup is often the result of a local-global transition (utter the words "gerbe, torsor" here).

Pseudo-aside: If that *is* the case here, then it'll be easier to construct a bijection using this than to do it directly! Why? Here's an answer by analogy: Suppose $\mathscr{L}, \mathscr{M}$ are two line bundles (or more generally, principle G-bundles). If we have a natural construction for a map of their global sections, then that should immediately give us a map of their sheaves of sections $\Gamma(-,\mathscr{L}) \to \Gamma(-,\mathscr{M})$. And conversely, having a map of their sheaves of sections would certainly give us a map of global sections. But the latter is actually "easier" to construct: Since sheaf-Hom is a sheaf, we can construct the map locally so long as we check that it's natural enough to glue. Then, we can take advantage of extra structure that might not be available globally (e.g., trivializations).

Pessimism+Idea: For liftings the local-global nature is clear (they form a stack $\boxed{\mathrm{Lift}(X'; \widetilde{S}/S)}$ on X'_{Zar}). For decompositions, something is wrong with our setup:First of all, there doesn't seem to be a convenient place to find $H^0(X', T_{X'/S})$ (automorphisms of a map?). Secondly, there's no good way to glue maps in the derived category and so no good way to set up a local-global problem. So, how do we formulate a local-global problem whose global solutions on X' are precisely the decompositions of $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$?

While maps in the derived category don't glue, there *is* a way to "glue" actual chain maps (with suitable chain homotopies) to an element of the derived category (using Čech coverings). Even better, $H^1 \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S} \cong \Omega^1_{X'/S}$ is *locally free* and thus locally *projective*: So, for $U \subset X$ small enough (e.g., affine) $\mathrm{Hom}_{K(U)}(\Omega^1_{X'/S}, -) = \mathrm{Hom}_{D(U)}(\Omega^1_{X'/S}, -)$: That is, every map out of it in the derived category (e.g., our decomposition maps) is representable by an actual chain map (well-defined up to homotopy class). So, every decomposing map will arise in this way: by gluing honest chain maps via homotopies. This lets us set up a suitable local-global problem whose global objects will still be the decompositions of $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$: Take the prestack on X'_{Zar}

$$\mathrm{SDec}'(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}) : U' \mapsto \left\{ \begin{matrix} \text{Strict decomposition maps } \Omega^1_{X'/S}\big|_{U'} \to \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}\big|_{U'} \\ \text{Chain homotopies as morphisms} \end{matrix} \right\}$$

and define $\boxed{\mathrm{SDec}(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S})}$ to be the associated stack. One can verify (more-or-less by the above argument) that the isomorphism classes of global sections of $\mathrm{SDec}(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S})$ are in bijection with decompositions.