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Abstract. Suppose (M, f) is an LG model. There are ways in which the 2-periodic dg-category of matrix factorization $\mathrm{MF}(M, f)$ "wants to be" like (a 2-periodicization of) the dg-category $\mathrm{Perf}(\mathrm{crit}(f))$ of perfect complexes on the critical locus, although this is not literally true. In this note we show an amusing sense in which this is true: If (M, f) is a non-degenerate quadratic bundle over a scheme, then the statement holds at the level of Hochschild cohomology (and almost Hochschild homology).

1. Introduction

Notation 1.0.1. For the duration of this document: X is a smooth scheme over a characteristic-zero field k, $\mathcal{Q} \to X$ a vector bundle, and $q\colon \mathcal{Q} \to \mathbb{A}^1$ is a non-degenerate quadratic form on $\mathcal{Q}$. Since q is non-degenerate, the critical locus $\mathrm{crit}(q) = X$. We will be interested in the k-linear dg-category $\mathrm{Perf}(X)$ of perfect complexes on X, and the $k(\!(\beta)\!)$-linear (homological $\deg \beta = -2$) dg-category $\mathrm{MF}(\mathcal{Q}, q)$ of matrix factorizations for q on $\mathcal{Q}$.

Our computation will be based on two results from [P]:

Theorem 1.0.2 ([P, Theorem 9.1.7(ii)])**.** *Suppose $\mathcal{Q}$ is a* metabolic *quadratic bundle, i.e., there is a sub-vector bundle $\mathcal{L} \subset \mathcal{Q}$ such that $\mathcal{L} = \mathcal{L}^{\perp}$. Then, tensor product with $\mathcal{O}_{\mathcal{L}}$ induces an equivalence*

$$- \otimes_{\mathcal{O}_X(\!(\beta)\!)} \mathcal{O}_{\mathcal{L}}\colon\ \mathrm{Perf}(X)(\!(\beta)\!) \xrightarrow{\sim} \mathrm{MF}(\mathcal{Q}, q)$$

Theorem 1.0.3 ([P, Theorem 8.2.6])**.** *Since 0 is the only critical value of q, there are $k(\!(\beta)\!)$-linear equivalences*

$$\mathbf{HH}^{\bullet}_{k(\!(\beta)\!)}(\mathrm{MF}(\mathcal{Q}, q)) = \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}(\mathcal{Q}^2, -q \boxplus q)}\left(\overline{\Delta}_* \omega_{\mathcal{Q}}, \overline{\Delta}_* \omega_{\mathcal{Q}}\right) = \mathrm{RHom}^{\otimes k}_{\mathrm{QC}(\mathcal{Q}^2)}\left(\Delta_* \omega_{\mathcal{Q}}, \Delta_* \omega_{\mathcal{Q}}\right)^{\mathrm{Tate}}$$

$$\mathbf{HH}_{\bullet}^{k(\!(\beta)\!)}(\mathrm{MF}(\mathcal{Q}, q)) = \mathrm{RHom}^{\otimes k(\!(\beta)\!)}_{\mathrm{MF}(\mathcal{Q}^2, -q \boxplus q)}\left(\overline{\Delta}_* \mathcal{O}_{\mathcal{Q}}, \overline{\Delta}_* \omega_{\mathcal{Q}}\right) = \mathrm{RHom}^{\otimes k}_{\mathrm{QC}(\mathcal{Q}^2)}\left(\Delta_* \mathcal{O}_{\mathcal{Q}}, \Delta_* \omega_{\mathcal{Q}}\right)^{\mathrm{Tate}}$$

where $\overline{\Delta}$ is the factorization of the diagoanl $\Delta\colon \mathcal{Q} \to \mathcal{Q}^2$ through the zero locus of the superpotential $-q \boxplus q$.

Using these, we will conclude

Theorem 1.0.4. *There is an equivalence*

$$\mathbf{HH}^{\bullet}_{k(\!(\beta)\!)}(\mathrm{MF}(\mathcal{Q}, q)) = \mathbf{HH}^{\bullet}_{k}(\mathrm{Perf}(X)) \otimes_k k(\!(\beta)\!)$$

while the analogous statement for $\mathbf{HH}_{\bullet}$ requires taking coefficients in the (bimodule corresponding to) the line bundle $\det \mathcal{Q}^{\vee}[d]$

$$\mathbf{HH}_{\bullet}^{k(\!(\beta)\!)}(\mathrm{MF}(\mathcal{Q}, q)) = \mathbf{HH}_{\bullet}^{k}\left(\mathrm{Perf}(X), \det \mathcal{Q}^{\vee}\right) \otimes_k k(\!(\beta)\!)$$

Remark 1.0.5.
- The right hand sides admit evident descriptions via HKR. Having to twist one of the two sides is to be expected from considerations of when $\mathrm{Perf}(X)$ and $\mathrm{MF}(\mathcal{Q}, q)$ should be Calabi-Yau: $\det \mathcal{Q}^{\vee}[d]$ pulls back to the relative dualizing bundle $\omega_{\mathcal{Q}/X}$.
- The identification of $\mathbf{HH}^{\bullet}$ also follows formally from a Corollary of [P, Theorem 9.1.7] included in [P]: that $\mathrm{MF}(\mathcal{Q}, q)$ is an invertible $\mathrm{Perf}(X)(\!(\beta)\!)^{\otimes}$-module category.
- Outside of the "massive" (i.e., non-degenerate quadratic) case, the above Theorem rapidly ceases to be true.[1]

[1] e.g., for $x^3\colon \mathbb{A}^1 \to \mathbb{A}^1$ one can check that $\pi_* \mathbf{HH}^{\bullet}_{k(\!(\beta)\!)}(\mathrm{MF}(\mathbb{A}^1, x^3))$ identifies with the 2-periodization of the Jacob ring $k[x]/(3x^2)$ while $\pi_* \mathbf{HH}^{\bullet}_{k}(\mathrm{Perf}(\mathrm{crit}(x^3)))$ is $k[x]/x^2$ in degree 0 and k in all negative degrees.↑

2. Proof

2.0.6. Equip $\mathcal{Q} \times_X \mathcal{Q}$ and $\mathcal{Q}^2$ with the superpotential $-q \boxplus q$, and let $(\mathcal{Q} \times_X \mathcal{Q})_0$ and $(\mathcal{Q}^2)_0$ denote the respective fibers over zero. Consider the commutative diagram, with Cartesian squares

$$\begin{array}{ccccccc}
\mathcal{Q} & \xrightarrow{\widetilde{\Delta}} & (\mathcal{Q} \times_X \mathcal{Q})_0 & \xrightarrow{i} & \mathcal{Q} \times_X \mathcal{Q} & \xrightarrow{p} & X \\
& \searrow^{\overline{\Delta}} & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Delta_X} \\
& & (\mathcal{Q}^2)_0 & \xrightarrow{j} & \mathcal{Q}^2 & \xrightarrow{q} & X^2
\end{array}$$

where $\widetilde{\Delta}$ is the indicated factorization of the relative diagonal $\Delta \colon \mathcal{Q} \to \mathcal{Q} \times_X \mathcal{Q}$ through the zero-locus of the superpotential.

2.0.7. Note that $-q \boxplus q \colon \mathcal{Q} \times_X \mathcal{Q} \to \mathbb{A}^1$ is a non-degenerate quadratic bundle on X: It is the *hyperbolic* quadratic bundle normally denoted $\overline{\mathcal{Q}} \perp \mathcal{Q}$. The diagonal $\Delta_{\mathcal{Q}}$ is a Lagrangian subspace, so that $\widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}}$ generates the equivalence of Theorem 1.0.2. Let $d = \dim_X \mathcal{Q}$ be the fiber dimension of $\mathcal{Q}$ and $\omega' = \mathbb{D}((\det \mathcal{Q})[-d]) = \omega_X \otimes \det \mathcal{Q}^\vee [d]$, so that the usual determinant formula gives $\omega_{\mathcal{Q}} = \omega'|_{\mathcal{Q}}$; by the projection formula, $\widetilde{\Delta}_* \omega_{\mathcal{Q}} = \omega' \otimes_{\mathcal{O}_X} \widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}}$. Since h_0 is a base-change of Δ_X and X is smooth, it is of finite Tor-dimension.

2.0.8. By Theorem 1.0.2, Theorem 1.0.3, and plenty of base-change

$$\begin{aligned}
\mathbf{HH}^\bullet_{k((\beta))}(\mathrm{MF}(\mathcal{Q}, q)) &= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q}^2, -q\boxplus q)} \left((h_0)_* \widetilde{\Delta}_* \omega_{\mathcal{Q}}, (h_0)_* \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left((h_0)^* (h_0)_* \widetilde{\Delta}_* \omega_{\mathcal{Q}}, \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left(\widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_{(\mathcal{Q} \times_X \mathcal{Q})_0}} (h_0)^* (h_0)_* \mathcal{O}_{(\mathcal{Q} \times_X \mathcal{Q})_0}, \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left(\widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_X} (\omega' \otimes_{\mathcal{O}_X} (\Delta_X)^* (\Delta_X)_* \mathcal{O}_X), \widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_X} \omega' \right) \\
&= \mathrm{RHom}^{\otimes k}_{\mathrm{Perf}(X)} (\omega' \otimes_{\mathcal{O}_X} (\Delta_X)^* (\Delta_X)_* \mathcal{O}_X, \omega') \otimes_k k((\beta)) \\
&= \mathrm{RHom}^{\otimes k}_{\mathrm{Perf}(X)} ((\Delta_X)^* (\Delta_X)_* \mathcal{O}_X, \mathcal{O}_X) \otimes_k k((\beta)) \\
&= \mathbf{HH}^\bullet_k (\mathrm{Perf}(X)) \otimes_k k((\beta))
\end{aligned}$$

2.0.9. And the analogous computation for $\mathbf{HH}_\bullet$:

$$\begin{aligned}
\mathbf{HH}_\bullet^{k((\beta))}(\mathrm{MF}(\mathcal{Q}, q)) &= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q}^2, -q\boxplus q)} \left((h_0)_* \widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}}, (h_0)_* \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left((h_0)^* (h_0)_* \widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}}, \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left(\widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_{(\mathcal{Q} \times_X \mathcal{Q})_0}} (h_0)^* (h_0)_* \mathcal{O}_{(\mathcal{Q} \times_X \mathcal{Q})_0}, \widetilde{\Delta}_* \omega_{\mathcal{Q}} \right) \\
&= \mathrm{RHom}^{\otimes k((\beta))}_{\mathrm{MF}(\mathcal{Q} \times_X \mathcal{Q}, -q\boxplus q)} \left(\widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_X} (\Delta_X)^* (\Delta_X)_* \mathcal{O}_X, \widetilde{\Delta}_* \mathcal{O}_{\mathcal{Q}} \otimes_{\mathcal{O}_X} \omega' \right) \\
&= \mathrm{RHom}^{\otimes k}_{\mathrm{Perf}(X)} ((\Delta_X)^* (\Delta_X)_* \mathcal{O}_X, \omega') \otimes_k k((\beta)) \\
&= \mathrm{RHom}^{\otimes k}_{\mathrm{Perf}(X)} \left((\Delta_X)_* \mathcal{O}_X, (\Delta_X)_* \left(\omega_X \otimes \det \mathcal{Q}^\vee [d] \right) \right) \otimes_k k((\beta)) \\
&= \mathbf{HH}_\bullet^k \left(\mathrm{Perf}(X), \det \mathcal{Q}^\vee [d] \right) \otimes_k k((\beta))
\end{aligned}$$

References

[P] Anatoly Preygel, *Thom-Sebastiani & duality for matrix factorizations*, arXiv e-print (January 2011), available at arxiv:1101.5834.