KOSZUL FUN

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1. p-local Pontryagin duality

1.1. **Setup.**

1.1.1. Setup: R a Noetherian regular local ring, $\mathfrak{m}$ maximal ideal, $k = R/\mathfrak{m}$. Choose $f_1, \dots, f_r \in \mathfrak{m}$ a minimal set of generators (i.e., their images are a k-basis for $\mathfrak{m}/\mathfrak{m}^2$).

1.1.2. Set

$$\mathscr{E} = \mathrm{RHom}_R(k, k)$$

Then, $\mathscr{E}$ is an associative (dg) R-algebra, and k is a left $\mathscr{E}$-module.

1.1.3. Set

$$k^\vee = \mathrm{RHom}_R(k, R)$$

It is a right $\mathscr{E}$-module, and our goal will be to compute the R-module

$$k^\vee \overset{L}{\otimes}_{\mathscr{E}} k.$$

1.1.4. Recall the Koszul complex for $k = R/\mathfrak{m}$ over R:

$$K = R[\epsilon_1, \dots, \epsilon_r] \qquad |\epsilon_i| = 1,\ d\epsilon_i = f_i$$

Or more verbosely (but not describing the differential)

$$K_\bullet = \left[\underbrace{R \cdot \epsilon_1 \wedge \cdots \wedge \epsilon_r}_{\text{deg. } r} \longrightarrow \cdots \longrightarrow \oplus_{1 \le i < j \le d} R \cdot \epsilon_i \wedge \epsilon_j \longrightarrow \oplus_{i=1}^d R \cdot \epsilon_i \longrightarrow \underbrace{R \cdot 1}_{\text{deg. } 0} \right]$$

The map $K \to K/(f_1, \dots, f_r) = k$, $1 \mapsto 1$, is a quasi-isomorphism. (Pf: Induction on "d", regular sequence, ...)

1.1.5. The "Hodge star" makes the Koszul complex K self-dual up to a degree shift: That is, $*u$ is characterized by $u \wedge *u = \epsilon_1 \wedge \cdots \wedge \epsilon_r$. It consists of free R-modules so that

$$\mathrm{RHom}_R(K, R) = \mathrm{Hom}_R(K, R) = \left[\underbrace{R \cdot 1^\vee}_{\text{deg. } 0} \longrightarrow \cdots \longrightarrow \underbrace{R \cdot (\epsilon_1 \wedge \cdots \wedge \epsilon_r)^\vee}_{\text{deg. } -r} \right]$$

$$\simeq \left[\underbrace{R \cdot *1}_{\text{deg. } 0} \longrightarrow \cdots \longrightarrow \underbrace{R \cdot *(\epsilon_1 \wedge \cdots \wedge \epsilon_r)}_{\text{deg. } -r} \right] = K[r].$$

Consequently $k^\vee = k[r]$.

1.1.6. In particular, $k[r]$ (and so k) *inherits a right $\mathscr{E}$-module structure from that of $k^\vee$.* (Note that this, at least a priori, depends on the choice of $f_1, \dots, f_r$.) This lets us re-phrase our goal as computing

$$k \overset{L}{\otimes}_{\mathscr{E}} k \left(\simeq (k^\vee \overset{L}{\otimes}_{\mathscr{E}} k)[-r] \right)$$

Example 1.1.7. Set $R = \mathbb{Z}_{(p)}$, $\mathfrak{m} = p\mathbb{Z}_{(p)}$, $k = \mathbb{F}_p$; $r = 1$, and we can take $f_1 = p$. With the above identifications, we will show that $k \overset{L}{\otimes}_{\mathscr{E}} k \simeq \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}$ The natural "evaluation" map

$$k \overset{L}{\otimes}_{\mathscr{E}} k \simeq (k^\vee \overset{L}{\otimes}_{\mathscr{E}} k)[-1] = (\mathrm{RHom}_R(k, R) \overset{L}{\otimes}_{\mathscr{E}} R)[-1] \xrightarrow{\mathrm{ev}} R[-1]$$

will correspond to the "boundary map" $\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}[-1]$ associated to the exact triangle

$$\mathbb{Z}_{(p)} \to \mathbb{Q}_{(p)} \to \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}$$

1.2. Explicit Computation for $R = \mathbb{Z}_{(p)}$.

1.2.1. It might be illustrative to do the computation of the above example explicitly. We'll give some elements names:

$$K = \left[R \cdot a \xrightarrow{p} R \cdot 1\right]$$

(and using the magical sign rule)

$$K^\vee = \left[R \cdot 1^\vee \xrightarrow{-p} R \cdot a^\vee\right]$$

As R-complex,

$$\mathscr{E} = K \otimes_R K^\vee$$

is spanned by $a^\vee \otimes 1$ in degree -1, $a^\vee \otimes a$ and $1^\vee \otimes 1$ in degree 0, and $1^\vee \otimes a$ in degree 1. (With differentials determined by the tensor product rules). The dg algebra structure is given simply by "evaluation": $(v \otimes \lambda) \cdot (v' \otimes \lambda') = \lambda(v')(v \otimes \lambda')$. (Good thing R has no differential!)

1.2.2. I think I even got the signs right to, e.g., make $\mathscr{E}$ a dga. For instance,

$$d(1) = d(1 \otimes 1^\vee + a \otimes a^\vee) = -p(1 \otimes a^\vee) + p(1 \otimes a^\vee) = 0$$

Even trickier, set $\alpha = v \otimes \lambda$ and $\beta = v' \otimes \lambda'$. Then,

$$d(\alpha \cdot \beta) = \lambda(v') d\left(v \otimes \lambda'\right) = \lambda(v')\left(dv \otimes \lambda' + (-1)^{|v|} v \otimes d(\lambda')\right).$$

$$d(\alpha) \cdot \beta = \underbrace{\lambda(v')(dv \otimes \lambda')}_{\text{if } |v'| = -|\lambda|} + (-1)^{|v|} \underbrace{(d\lambda)(v')(v \otimes \lambda')}_{\text{if } |v'| = 1 - |\lambda|}$$

$$\alpha \cdot d(\beta) = \underbrace{\lambda(dv')(v \otimes \lambda')}_{\text{if } |v'| = 1 - |\lambda|} + (-1)^{|v'|} \underbrace{\lambda(v')(v \otimes d\lambda')}_{\text{if } |v'| = -|\lambda|}$$

So,

$$\begin{aligned} d(\alpha \cdot \beta) &= d(\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot d(\beta) \\ &= d(\alpha) \cdot \beta + (-1)^{|v| + |\lambda|} \alpha \cdot d(\beta) \\ &= \lambda(v')\left(dv \otimes \lambda' + (-1)^{|v| + |\lambda| + |v'|}(v \otimes d\lambda')\right) \\ &\quad + (-1)^{|v|}(v \otimes \lambda')\left((d\lambda)(v') + (-1)^{|\lambda|}\lambda(dv')\right) \end{aligned}$$

Since $\lambda(v') = 0$ unless $|\lambda| + |v'| = 0$, the first term is precisely $d(\alpha \cdot \beta)$. Meanwhile, since R is in a single degree

$$0 = d(\lambda(v')) = (d\lambda)(v') + (-1)^{|\lambda|}\lambda(dv').$$

(This was the reason for the "magical sign rule," forcing $d(1^\vee)(a) = -1^\vee(da) = -p$.)

1.2.3. We can work with the algebra structure on $\mathscr{E}$, and the $\mathscr{E}$-module structure on K, via "matrices." Namely, the following identifications itertwine the products and module structures

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \longrightarrow x(a \otimes a^\vee) + y(a \otimes 1^\vee) + z(1 \otimes a^\vee) + w(1 \otimes 1^\vee)$$

and

$$\begin{pmatrix} r \\ s \end{pmatrix} \longrightarrow r \cdot a + s \cdot 1$$

1.2.4. Now, let's resolve K as left $\mathscr{E}$-module: There's a surjection $\mathscr{E} \twoheadrightarrow K$, $\alpha \mapsto \alpha \cdot 1$. In matrices this is

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \longrightarrow \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y \\ w \end{pmatrix}$$

so the kernel of the surjection consists of matrices whose second column is zero (i.e., no $a \otimes 1^\vee$ or $1 \otimes 1^\vee$ component).

Let $t = 1 \otimes a^\vee$, and consider right multiplication $\cdot t \colon \mathscr{E}[1] \to \mathscr{E}$. Then, $\mathrm{im}(\cdot t) = \ker(\cdot t)$ is also the matrices with second column zero. In other words,

$$\mathrm{Tot}\left[\cdots \xrightarrow{\cdot t} \mathscr{E}[3] \xrightarrow{\cdot t} \mathscr{E}[2] \xrightarrow{\cdot t} \mathscr{E}[1] \xrightarrow{\cdot t} \mathscr{E}\right] \to K$$

is a free resolution of K.

1.2.5. We use the previous resolution to compute $K^\vee \overset{L}{\otimes}_{\mathscr{E}} K$:

$$K^\vee \overset{L}{\otimes}_{\mathscr{E}} K = \mathrm{Tot}\left[\cdots \xrightarrow{\cdot t} K^\vee \xrightarrow{\cdot t} K^\vee \xrightarrow{\cdot t} K^\vee\right].$$

The right action of $t = 1 \otimes a^\vee$ on $K^\vee$ is easy to compute: $(1^\vee) \cdot t = a^\vee$ while $(a^\vee) \cdot t = 0$. The double complex we're totalizing looks like

$$\begin{array}{ccccc}
 & & & & R \cdot 1^\vee \\
 & & & & \downarrow{\scriptstyle -p} \\
 & & R \cdot 1^\vee & \xrightarrow{=} & R \cdot a^\vee \\
 & & \downarrow{\scriptstyle -p} & & \\
R \cdot 1^\vee & \xrightarrow{=} & R \cdot a^\vee & & \\
\downarrow{\scriptstyle -p} & & & & \\
R \cdot a^\vee & & & &
\end{array}
\qquad \cdots$$

with all the $R \cdot 1^\vee$ terms in total degree 0, and all the $R \cdot a^\vee$ in total degree -1. In other words, its totalization is the 2-term complex

$$\overbrace{\bigoplus_{i \geq 0} R}^{\text{deg. } 0} \xrightarrow{d} \overbrace{\bigoplus_{i \geq 0} R}^{\text{deg. } -1}$$

where

$$d(x_0, x_1, x_2, \cdots) = (x_1 - px_0, x_2 - px_1, x_3 - px_2, \cdots).$$

1.2.6. There should be some sort of standard telescope-type Lemma that allows us to immediately say that the above complex is the hofiber of

$$R \to \varinjlim \left\{ R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} \cdots \right\}$$

Since

$$\mathbb{Q}_{(p)} = \varinjlim \left\{ R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} \cdots \right\}$$

this will be the hofiber of $\mathbb{Z}_{(p)} \to \mathbb{Q}_{(p)}$, i.e., $\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}[1]$. So, $k^\vee \overset{L}{\otimes}_{\mathscr{E}} k \simeq \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}[1]$ and $k \overset{L}{\otimes}_{\mathscr{E}} k \simeq \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}$ as claimed!

1.2.7. En lieu of thinking about the above standard Lemma, we can just explicitly write down a quasi-isomorphism. Consider

$$\begin{array}{ccc}
\bigoplus_{i \geq 0} R & \xrightarrow{x_0} & \mathbb{Z}_p \\
\downarrow{\scriptstyle d} & & \downarrow \\
\bigoplus_{i \geq 0} R & \xrightarrow[\Phi]{} & \mathbb{Q}_p
\end{array}$$

where x_0 denotes projection to the first component and

$$\Phi(y_0, y_1, \ldots) = -\sum_{i\geq 0} \frac{1}{p^{i+1}} y_i$$

is concocted so that the diagram commutes. We will be done if we can show that this is a quasi-isomorphism.

Filter the LHS by declaring F_N to be the subcomplex where both direct sums are over $0 \leq i < N$; filter the RHS by declaring F_N to be $\mathbb{Z}_p \to 1/p^N \mathbb{Z}_p$. We verify that x_0 and Φ are compatible with filtrations, so it is enough to prove that this is a quasi-isomorphism on each filtered piece. On each filtered piece, writing down matrices verifies that both differentials are injective, have cokernels isomorphic to $\mathbb{Z}/p^N$, and the map on cokernels is an isomorphism.

1.3. **Conceptual computation.**

1.3.1. The explicit computation of the previous section can of course be made to work in general. Instead, we give a more conceptual reformulation.

1.3.2. We may regard k as an R-$\mathscr{E}^{\mathrm{op}}$ bimodule (i.e., a left R-module and right $\mathscr{E}$-module compatibly). It therefore determines an adjoint pair

$$\mathcal{L}\colon \mathscr{E}^{\mathrm{op}}\text{-mod} \rightleftarrows R\text{-mod}\colon \mathcal{R}$$

$$\mathcal{L}(\mathscr{F}) = \mathscr{F} \overset{L}{\otimes}_{\mathscr{E}} k \qquad \text{and} \qquad \mathcal{R}(M) = \mathrm{RHom}_R(k, M).$$

The functor $\mathcal{L}$ is essentially characterized by $\mathcal{L}(\mathscr{E}) = k$ and preserving colimits. The Koszul complex of k demonstrates that it is compact ("small"), so $\mathcal{R}$ *also preserves arbitrary colimits*. So understanding $\mathcal{R}(M)$ (resp., $\mathcal{L}(\mathcal{R}(M))$) for arbitrary $M \in R$-mod reduces (via colimits) to computing $\mathcal{R}(R)$ (resp., $\mathcal{L}(\mathcal{R}(R))$).

Note that

$$\mathcal{L}(\mathcal{R}(R)) = \mathrm{RHom}_R(k, R) \otimes_{\mathscr{E}} k$$

is exactly what we wanted to compute earlier.

1.3.3. Let i be the inclusion of (the underlying topological space of) $Z = \operatorname{Spec} k = \operatorname{Spec} R/\mathfrak{m}$ into $\operatorname{Spec} R$. There is an adjoint pair

$$i_*\colon R\text{-mod}_Z \rightleftarrows R\text{-mod}\colon i^!$$

where R-mod$_Z$ denotes the category of R-modules set-theoretically supported on Z.

1.3.4. The "conceptual claim" is that there is an equivalence of categories $R\text{-mod}_Z \simeq \mathscr{E}^{\mathrm{op}}$-mod intertwining these adjunctions. Then,

$$\mathcal{L}(\mathcal{R}(R)) = i_*(i^!(R))$$

Consider the triangles

$$R \to R[f_1^{-1}] \to R/(f_1^\infty)$$

$$R/(f_1^\infty) \to R/(f_1^\infty)[f_2^{-1}] \to R/(f_1^\infty, f_2^\infty)$$

and so on, until

$$R/(f_1^\infty, \ldots, f_{r-1}^\infty) \to R/(f_1^\infty, \ldots, f_{r-1}^\infty)[f_r^{-1}] \to R/(f_1^\infty, \ldots, f_r^\infty)$$

In each case, the middle term is supported off of Z, so $i^!$ of it is zero. Also, $R/(f_1^\infty, \ldots, f_r^\infty)$ is supported on Z so that $i_* \circ i^!$ of it is itself. Applying $i_* \circ i^!$ to the triangles we obtain

$$i_* i^! R \to 0 \to i_* i^! R/(f_1^\infty)$$

$$i_* i^! R/(f_1^\infty) \to 0 \to i_* i^! R/(f_1^\infty, f_2^\infty)$$

and so on, until

$$i_* i^! R/(f_1^\infty, \ldots, f_{r-1}^\infty) \to 0 \to R/(f_1^\infty, \ldots, f_r^\infty)$$

Tracing through, we obtain

$$\boxed{i_* i^! R \simeq R/(f_1^\infty, \ldots, f_r^\infty)[r]}$$

1.3.5. We mention how to prove the conceptual claim: It suffices to prove that $\mathcal{R}$ is essentially surjective, $\mathcal{L}$ is fully faithful, and identify the essential image of $\mathcal{L}$ with $R\text{-mod}_Z$. (Then, taking $\mathscr{E} \to k \in R\text{-mod}_Z$ specifies the equivalence. It intertwines i_* and $\mathcal{L}$, and the rest follows.)

Claim: The unit $\mathrm{id} \to \mathcal{R} \circ \mathcal{L}$ is an equivalence of functors. Certainly $\mathcal{R}(\mathcal{L}(\mathscr{E})) = \mathrm{RHom}_R(k, \mathscr{E} \overset{L}{\otimes}_{\mathscr{E}} k) = \mathscr{E}$, and we extend under colimits. This proves that $\mathcal{R}$ is essentially surjective, and that $\mathcal{L}$ is faithful.

Claim: If $M \in R\text{-mod}_Z$, then the counit $\mathcal{L}(\mathcal{R}(M)) \to M$ is an equivalence. Certainly $\mathcal{L}(\mathcal{R}(k)) = \mathrm{RHom}_R(k,k) \overset{L}{\otimes}_{\mathscr{E}} k = \mathscr{E} \overset{L}{\otimes}_{\mathscr{E}} k = k$, and we extend under colimits. This proves that the essential image of $\mathcal{L}$ contains $R\text{-mod}_Z$, and so is equal to it (since $R\text{-mod}_Z$ is closed under colimits and contains k). Since $\mathcal{R}$ is essentially surjective, this also proves that $\mathcal{L}$ is full.

1.3.6. We didn't actually need the "conceptual claim" to run the "conceptual proof." All we needed was that

- Suppose $M \in R\text{-mod}$ is such that $f_i : M \to M$ is an equivalence for some i. Then, $\mathcal{R}(M) = 0$. This follows directly, using e.g. the Koszul complex.
- Suppose $M \in R\text{-mod}_Z$. Then, $\mathcal{L}(\mathcal{R}(M)) = M$. This is the argument of the above claim.

Then, applying $\mathcal{L} \circ \mathcal{R}$ to the sequence of triangles yields the desired claim.