JUVITOP: HOCHSCHILD HOMOLOGY

ANATOLY PREYGEL
SEPTEMBER 23, 2009

1. Today

1.1. How Today Fits In. The point of the seminar is to study

$$\mathbf{THH} : \{\text{Assoc. alg in Spectra}\} \longrightarrow \{\text{Spectra}\}$$

Today we'll talk about

$$\mathbf{HH} : \{\text{Assoc. alg in } \mathbf{Ch}\} \longrightarrow \mathbf{Ch}$$

We'll see that a few of the nicer statements will work best over $\mathbb{Q}$. If we think that **THH** is the "right" object then we might think this is due to the following phenomenon: If A is an algebra in $\mathbf{Ch}_{\mathbb{Z}}$, then[1] $H(A)$ is an algebra in spectra but $H(\mathbf{HH}(A)) \neq \mathbf{THH}(H(A))$ in general. If, however, A is an algebra in $\mathbf{Ch}_{\mathbb{Q}}$, then $H(\mathbf{HH}(A)) = \mathbf{THH}(H(A))$! So (algebraic) Hochschild homology is good enough to pick out the "rational" part of THH. In effect, this is because the classical fact $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ extends to the spectra-level statement $H\mathbb{Q} \wedge H\mathbb{Q} = H\mathbb{Q}$.

Exercise 1.1.1. Verify $H\mathbb{Q} \wedge H\mathbb{Q} = H\mathbb{Q}$, by, e.g., computing (stable) rational homology of rational Eilenberg-MacLane spaces $\lim_m H_{\bullet+m}(K(\mathbb{Q}, m), \mathbb{Q})$. Show that $H\mathbb{Z} \wedge H\mathbb{Z} \neq H\mathbb{Z}$, by, e.g., exhibiting a non-trivial element of $\lim_m H_{\bullet+m}(K(\mathbb{Z}, m), \mathbb{Z})$ for $\bullet \neq 0$.

1.2. Goals for Today. We have two main goals for today:

- "Examples": Loop spaces; HKR and relation to differential theory.
- S^1 action on $\mathbf{HH}(A)$: What it is; how it lets you define $\mathbf{HC}, \mathbf{HP}, \mathbf{HC}^-$; $\mathbf{HP}$ as De Rham cohomology; trace map factors $K \to \mathbf{HC}^- \to \mathbf{HH}$.

1.3. Reminders.

$$\mathbf{HH}(A) \stackrel{\text{def}}{=} A \overset{L}{\underset{A \overset{L}{\otimes}_{\mathbb{Z}} A^{\text{op}}}{\otimes}} A$$

As always this can be computed by the usual bar complex $B(A, A \otimes A^{\text{op}}, A)$. Can also compute this using a "better" resolution of A as $A \otimes A^{\text{op}}$-module, giving rise to the "cyclic bar complex":

$$= \left| A \leftleftarrows A^{\otimes 2} \,\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} A^{\otimes 3} \,\substack{\longrightarrow \\ \longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow} \cdots \right|$$

Here, $\mathbb{Z}/(n+1)\mathbb{Z}$ acts on the n^{th} space of this simplicial set. It was an observation of Alain Connes that the Hochschild differential is well-behaved with respect to this action and so one can fruitfully mix Hoschild homology with group homology of the cyclic groups. This'll give rise to the circle action!

[1] For a chain complex A, $H(A)$ will denote the Eilenberg-MacLane spectrum↑

2. "Examples"

We have the following table of computations. In the first column X is either a manifold (in which case **HH** has to be computed using topological tensor products so that $A \otimes A^{\mathrm{op}} = C^\infty(M^2)$) or a regular affine scheme over $\mathbb{Q}$. In the second example, M is a connected space (assumed simply connected for the $A = C^*(M)$ case).

	$A = C^\infty(X)$, or $A = \Gamma(X, \mathcal{O}_X)$	$A = C_*(\Omega M)$, resp., $A = C^*(M)$	
$HH_k(A)$	Ω^k_A	$H_k(LM)$, resp. $H^*(LM)$	
S^1-action	De Rham differential	Loop-rotation actions	
$HC_k(A)$	...	$H^{S^1}_k(LM)$,...	
$HP_k(A)$	$\prod_n H^{k+2n}_{\mathrm{dR}}(X)$	$H^{S^1}_k(LM)[u^{-1}]$	...

2.1. Hochschild-Kostant-Rosenberg – First Take.

In this section A will be a regular commutative ring over $\mathbb{Q}$ (or $A = C^\infty(X)$ for a manifold, with the provision that $X \otimes X^{\mathrm{op}}$ is interpreted as $C^\infty(X^2)$).

2.1.1. The geometric idea is to think of

$$\mathbf{HH}(A) = A \otimes_{A \otimes A^{\mathrm{op}}} A$$

as a "derived self-intersection of the diagonal" and to resolve the diagonal using the identification of the normal bundle of $X \subset X^2$ and the tangent bundle of X.

2.1.2. As a warm-up, we do the cases of $HH_0(A), HH_1(A)$ (this does not seem to need A to be regular!). Indeed if we identify $A \otimes A$ with $\Gamma(X^2, \mathcal{O}^2_X)$ (or $C^\infty(M^2)$), then A fits into a short exact sequence

$$0 \to I = \{\text{Functions vanishing on the diagonal}\} \to A \otimes A \longrightarrow A \to 0$$

Those familiar with the algebraic framework will recall the definition

$$\Omega^1_A \stackrel{\mathrm{def}}{=} I \otimes_{A \otimes A} A = I/I^2.$$

But the long exact sequence of $\mathrm{Tor}^*(-, A)$ includes the segment

$$\mathrm{Tor}^1(A \otimes A, A) = 0 \to \mathrm{Tor}^1(A, A) \to \mathrm{Tor}^0(I, A) \to \mathrm{Tor}^0(A \otimes A, A) \to \mathrm{Tor}^0(A, A) \to 0$$

The map $\mathrm{Tor}^0(I, A) \to \mathrm{Tor}^0(A \otimes A, A)$ (i.e., $I/I^2 \to A$) is the zero map, so that we obtain

$$HH_0(A) = (A \otimes A)/I = A \qquad \text{and} \qquad HH_1(A) = \mathrm{Tor}^1(A, A) = \mathrm{Tor}^0(I, A) = \Omega^1_A.$$

In general we need to do more work:

Proposition 2.1.3 (Hochschild-Kostant-Rosenberg)**.** *Suppose A is a regular commutative ring of characteristic zero. Then,*

$$HH_i(A) = \Omega^i = \bigwedge^i \Omega^1 = \textit{Module of Kahler } i\textit{-forms}$$

for all $i \geq 0$.

Proof. If A is regular then $A \otimes A \to A$ is a "locally complete intersection" ring map. This corresponds to the fact that X is smooth iff, locally near the diagonal $X \subset X^2$, the diagonal looks like a regular intersection of $n = \dim A$ sections of a vector bundle V. If this vector bundle were defined on all of X^2, not just near X, then we could use its higher exterior powers (with a regular set of sections) to give a resolution (the "Koszul resolution") of A as $A \otimes A$-module. The general case requires a bit more work.

We'll indicate the proof in the special case $A = k[x_1, \ldots, x_n]$. Set $R = A \otimes A = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, so that

$$I = (x_1 - y_1, \cdots, x_n - y_n)R \subset R \quad \text{and the vector bundle is the trivial bundle} \quad V = \bigoplus e_i R$$

The resolution we will want is

$$\bigwedge^n V \to \cdots \to \bigwedge^2 V \to \bigwedge^1 V \to \bigwedge^0 V = R \twoheadrightarrow A$$

Let's work out the first few bits: The surjection $V \twoheadrightarrow I = \ker(R \to A)$ is given by $e_i \mapsto x_i - y_i$. What is the kernel of this surjection? One can check that it's generated by elements of the form $(x_i - y_i)e_j - (x_j - y_j)e_i$, which we will regard as the image of $e_i \wedge e_j \in \bigwedge^2 V$. More generally, we can define a differential by

$$d(e_{i_0} \wedge \cdots \wedge e_{i_k}) = \sum (-1)^j (x_{i_j} - y_{i_j}) \cdots \wedge \widehat{e_{i_j}} \wedge \cdots$$

and prove that this gives a resolution of $A = R/I$ as R-module. □

2.2. **Loop Spaces.**

2.2.1. Nick will talk about this in more detail on October 28th (in *babytop*, but in the usual *juvitop* time). In the meantime, I can say something vague about how this is consistent with the previous example: Thinking of $C_*(\Omega X)$ as an algebraic manifestation of the homotopy type of X (think of ΩX as equivalent to the path groupoid of X), one might guess that "the derived self-intersection of the diagonal $X \subset X^2$" would relate to the homotopy fiber product $X \overset{h}{\times}_{X^2} X$.

2.2.2. What precisely *is* this homotopy pullback? It can be explicitly realized as the space of paths $\gamma : I \to X^2$ starting and ending on the diagonal. This is the same as pairs of paths $\gamma_1, \gamma_2 : I \to X$ satisfying $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. But *this* is nothing but the space of loops in X, so that

$$X \overset{h}{\times}_{X^2} X = LX \qquad \text{and} \qquad \mathbf{HH}(C_*(\Omega X)) \overset{\text{Claim}}{=} C_*(X \overset{h}{\times}_{X^2} X) = C_*(LX).$$

3. Circle Actions

3.1. **Complexes with S^1-action.** For this talk we'll need a baby version of "Borel S^1-equivariant ($H\mathbb{Z}$-)spectra", which we can define by hand in the context of dg-algebra:

Definition 3.1.1. Let **Ch** be the category of chain complexes of abelian groups: we use homological grading, i.e., differentials are of degree -1.
Define the category of "complexes with S^1-action" S^1-**Ch** to be the category of dg-modules over the graded algebra $C_*(S^1) \simeq H_*(S^1) = \mathbb{Z}[\epsilon]/(\epsilon^2)$, $\deg \epsilon = 1$. Explicitly, this is just a triple $(V_\bullet, d, \epsilon)$ consisting of a chain complex $(V_\bullet, d)$ together with a degree $+1$ chain map $\epsilon : V_\bullet \to V_\bullet[1]$. For this reason, these are sometimes called *mixed complexes*.

Construction 3.1.2. Suppose $V = (V_\bullet, d, \epsilon) \in S^1$-**Ch**. We have the "point with trivial action" $\mathbb{Z} = C_*(\mathrm{pt}) \in S^1$-**Ch**. We can define complexes

$$V_{hS^1} = V \overset{L}{\otimes}_{H_*(S^1)} \mathbb{Z}$$

$$V^{hS^1} = \mathrm{RHom}_{H_*(S^1)}(\mathbb{Z}, V)$$

These are modules over

$$\mathbb{Z}^{hS^1} = \mathrm{RHom}_{C_*(S^1)}(\mathbb{Z}, \mathbb{Z}) = \prod_* C^{-*}(BS^1) \simeq \mathbb{Z}[[u]], \quad \deg u = -2$$

and we may also look at the resulting 2-periodic version

$$V^{\mathrm{Tate}} = V^{hS^1}[u^{-1}] = \varinjlim \left\{ V^{hS^1} \xleftarrow{u} V^{hS^1} \xleftarrow{u} V^{hS^1} \xleftarrow{u} \cdots \right\}$$

There are natural maps $V^{hS^1} \to V \to V_{hS^1}$, and a Gysin-type map $\epsilon : V_{hS^1}[-1] \to V^{hS^1}$ which classifies a distinguished triangle $V^{hS^1} \to V^{\mathrm{Tate}} \to V_{hS^1}[-2]$.

3.1.3. More explicitly we have a reasonable resolution of $\mathbb{Z}$ as $H_*(S^1)$-module: It consists of $H_*(S^1) = \mathbb{Z}[\epsilon]/\epsilon^2$ in each degree, with all maps multiplication by ϵ. This gives rise to double complexes, whose totalizations ("limit" or "colimit" version, as appropriate) give explicit models

$$V_{hS^1} = (V[u^{-1}];\ d + u\epsilon \text{ with truncation in } \epsilon\text{-direction})$$

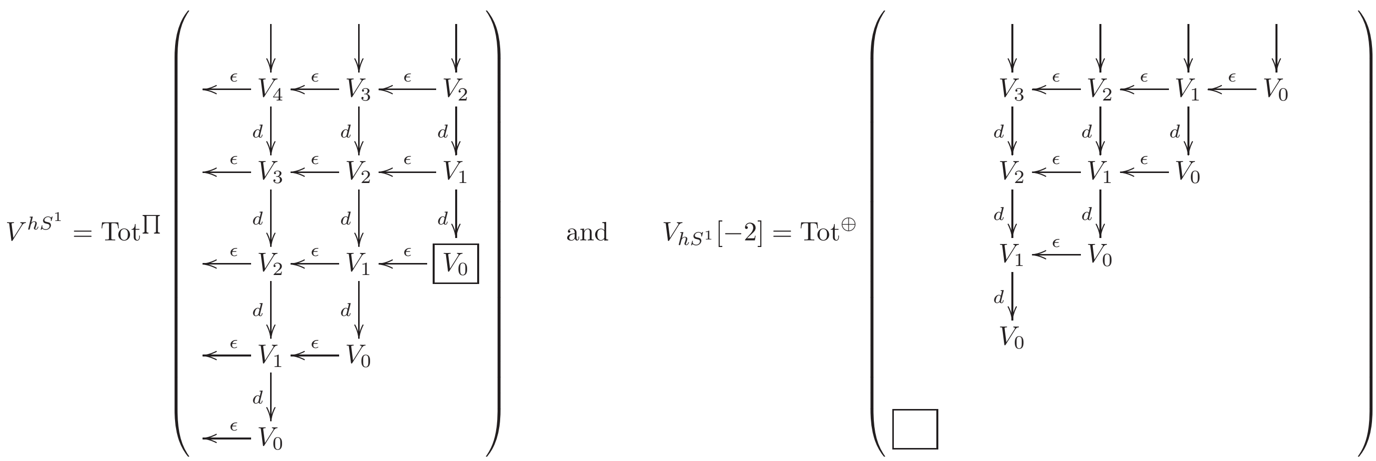
$$= \left(\bigoplus_{n \le 0} V[2n];\ d \text{ internal to } V,\ \epsilon \text{ shifting between copies (except where truncated)} \right)$$

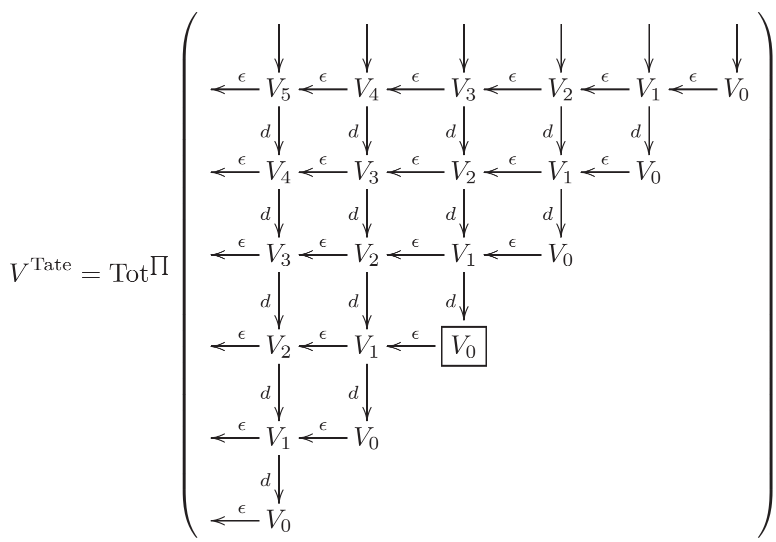
$$V^{hS^1} = (V[[u]];\, d + u\epsilon) = \left(\prod_{n\geq 0} V[2n];\, d \text{ internal to } V,\, \epsilon \text{ shifting between copies}\right)$$

$$V^{\mathrm{Tate}} = (V((u));\, d + u\epsilon) = \left(\prod_{n\in\mathbb{Z}} V[2n];\, d \text{ internal to } V,\, \epsilon \text{ shifting between copies}\right)$$

We've picked suggestive notation so that the $\mathbb{Z}[[u]]$-module structure is visible.

Remark 3.1.4. Suppose (for the sake of drawing things) that V lives in non-negative degrees. Then, the pictures to have in mind are (boxed entry is the $(0,0)$-bigraded piece):





So, V_{hS^1} consists of copies of V shifted in the "positive" direction (up and to the right), and with the ϵ differential truncated to the left. Meanwhile, V^{hS^1} consits of copies of V shifted in the "negative" direction (down and to the left) and the differentials don't have to be truncated. And, V^{Tate} consists of shifts in both directions (and with our choice of shift for V_{hS^1}), containing a copy of V^{hS^1} and mapping to V_{hS^1}.

Lemma 3.1.5 (Gysin Sequence)**.** *The above description of V_{hS^1} immediately gives rise to a Gysin exact triangle $V \to V_{hS^1} \to V_{hS^1}[-2]$, where the first map is an inclusion and the second map is the quotient. Passing to homotopy (i.e., homology of complexes) gives rise to a Gysin exact sequence*

$$\cdots \to \pi_n V \to \pi_n V_{hS^1} \to \pi_{n-2} V_{hS^1} \to \pi_{n-1} V \to \cdots$$

Remark 3.1.6. If X is a space with an S^1-action, then we can equip $V = C_*(X)$ with the structure of object in S^1-**Ch**. Then, $V_{hS^1} = C_*(X_{hS^1})$ and the natural spectral sequence $H_p(BS^1, H_q(X)) \Rightarrow H_{p+q}(X_{hS^1})$ is visible from the "obvious" (t-)filtration on V_{hS^1}. Making these identifications, the above Lemma recovers the usual Gysin sequence for $X \to X_{hS^1}$.

Remark 3.1.7. The name "Tate" in this context originally comes from the case of a finite cyclic group. The group homology and cohomology glue together into a 2-periodic cohomology theory: Tate cohomology. Consider the special case, of the above remark, $X = S^1$ with S^1 acting by the n^{th} power map. Then $X_{hS^1} = B\mathbb{Z}/n$, where $\mathbb{Z}/n \subset S^1$ is the n-torsion subgroup.

$$\begin{array}{ccc} S^1 & \longrightarrow & B\mathbb{Z}/n = (S_1)_{hS^1} \\ & & \downarrow \\ & & BS^1 \end{array}$$

Then, $V_{hS^1} = C_*(B\mathbb{Z}/n)$ is a complex computing group homology of $\mathbb{Z}/n$, $V^{hS^1} = \prod_* C^{-*}(B\mathbb{Z}/n)$ is (ignoring $\prod_*$ vs. $\bigoplus_*$) a complex computing group cohomology, and V^{Tate} is (ignoring $\prod_*$ vs $\bigoplus_*$) a complex computing what is called "Tate cohomology."

3.2. **Cyclic Structure on HH.** We can write down a concrete model for $\mathbf{HH}(A)$ as complex with S^1-action. For better or worse, I'd rather not just write down a formula. The derived tensor product description of $\mathbf{HH}(A)$ looks like it might admit a $\mathbb{Z}/2$-action, but not much more than that. The cleanest way to explain the S^1-action is via a specific model for *Borel S^1-equivariant spaces* called *cyclic sets*.

3.2.1. We give a brief overview of a few different reasonable notions of (the homotopy theory of) spaces with S^1-action:

– The strict sense: $S^1 \times X \to X$ satisfying the axioms for a group action. (Our definition of complex with S^1-action above is analogous to this.)
– The standard weak sense: Spaces over BS^1. Given X with an honest S^1-action, $X_{hS^1} \to BS^1$ is the resulting space over BS^1. Conversely, given a space $Y \to BS^1$, the action of ΩBS^1 on the homotopy fiber may be transfered to S^1.
– Our new sense (to be defined below): A cyclic set is roughly a simplicial set $X_\bullet$ with actions of $\mathbb{Z}/(n+1)$ on X_n for all n, suitably compatible. The claim is that the notion of a cyclic set $X_\bullet \in \mathbf{cSet}$ gives another model for this. (The cyclic bar complex that Nick wrote down last time is, appropriately enough, a cyclic object in chain complexes.)

More precisely, we have the following Proposition (proved in a 1985 paper of Dwyer-Hopkins-Kan):

Proposition 3.2.2. *There is an adjoint pair*

$$|\cdot| : \mathbf{cSet} \rightleftarrows S^1\text{-}Spaces : \mathrm{Sing}_{\Lambda_\bullet}$$

and a natural model structure on **cSet** *for which this is a Quillen equivalence with the Borel model structure on S^1-spaces. These are also equivalent, via homotopy fixed points and taking fibers, to* $\mathbf{sSet}/BS^1$.

Definition 3.2.3. Define the *cyclic category* with $\operatorname{ob}\Lambda^{\mathrm{op}} = \mathbb{N}_{\geq 0}$ and

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}([n],[m]) = \left\{\text{Homotopy classes of degree 1 increasing maps } \phi : S^1 \to S^1 \text{ s.t. } \phi(\mu_{n+1}) \subset \mu_{m+1}\right\}$$

Define the category of *cyclic sets*

$$\mathbf{cSet} = \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathbf{Set})$$

Remark 3.2.4. Note that

– Λ^{op} contains the simplicial category Δ^{op} (as ϕ s.t., $\phi(1) = 1$). In particular, we have a restriction functor $\mathbf{cSet} \to \mathbf{sSet}$.
– $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}([n],[n]) = \mathbb{Z}/(n+1)$ by rotation;
– And, every morphism of Λ^{op} is uniquely a composite of a morphism in Δ^{op} and a rotation. So, we may regard a cyclic set as a simplicial set together with a $\mathbb{Z}/(n+1)$-action on the n^{th} space, interacting in a certain specific way with the simplicial structure.
– Somewhat unexpectedly from this presentation, there is an equivalence $\Lambda^{\mathrm{op}} \simeq \Lambda$. In particular, Λ^{op} contains another copy of Δ^{op} (more than one due to automorphisms–but nevermind that)!

Construction 3.2.5. Suppose A is a (unital) dga with differential ∂. Then, the natural cyclic object we want is given by (using "bar" notation for the tensor)

$$[n] \longmapsto A^{\otimes(n+1)}$$
$$\{\phi : [n] \to [m]\} \longmapsto \left\{ A^{\otimes(n+1)} \ni (a_0|a_1|\cdots|a_n) \mapsto \left(\prod_{i\in\phi^{-1}(0)} a_i \middle| \cdots \middle| \prod_{i\in\phi^{-1}(m)} a_i \right) \in A^{\otimes(m+1)} \right\}$$

The associated complex with S^1-action is the geometric realization (=$\mathrm{Tot}^{\oplus}$) of a simplicial set we get from one of the "extra" copies of Δ^{op}. Explicitly: (my signs might be wrong in the graded case!)

$$\mathbf{HH}(A) = \left(\bigoplus_{n\geq 0} A^{\otimes(n+1)}[-n], d, B \right)$$

$$d\, \underbrace{(a_0|a_1|\cdots|a_n)}_{\deg = n+\sum_i \deg a_i} = \overbrace{\sum_{i=0}^{n-1} (-1)^{\varepsilon_i} (\cdots|a_{i-1}|a_i a_{i+1}|a_{i+1}|\cdots) + (-1)^{(\deg a_n+1)(\varepsilon_{n-1}+1)} (a_n a_0|a_1|..|a_{n-1})}^{d_1=\text{Simplicial differential}}$$
$$+ \underbrace{\sum_{i=0}^{n} (-1)^{\varepsilon_{i-1}} (\cdots|a_{i-1}|\partial a_i|a_{i+1}|\cdots)}_{d_0=\text{Internal differential of } A}$$

where $\varepsilon_i = i + 1 + \deg a_0 + \cdots + \deg a_i$ ($i \geq 0$), $\varepsilon_{-1} = 0$. The S^1-action ("ϵ") is provided by the *Connes B-operator*

$$B(a_0|\cdots|a_n) = \sum_{i=0}^{n} (-1)^{\epsilon_{i-1}(\epsilon_k - \epsilon_{i-1})} ((1|a_1|\cdots|a_n|a_0|\cdots|a_{i-1}) - (a_i|\cdots|a_n|a_0|\cdots|a_{i-1}|1))$$

3.2.6. One application of this machinery is the following: Attached to a topological group G is a natural set $\Gamma_\bullet G$, defined by formulas "suspiciously similar" to the above, and which has the property that $|\Gamma_\bullet G| \simeq L(BG)$. Applying this to $G = \Omega X$, we recover LX. This can lead to a proof of $\mathbf{HH}(C_*(\Omega X)) = C_*(LX)$, and compatibility with the cyclic structure shows compatibility of circle actions.

Definition 3.2.7. Define

The *cyclic homology* complex $\mathbf{HC} \stackrel{\text{def}}{=} \mathbf{HH}(A)_{hS^1}$

The *negative cyclic homology* complex $\mathbf{HC}^- \stackrel{\text{def}}{=} \mathbf{HH}(A)^{hS^1}$

The *periodic cyclic homology* complex $\mathbf{HP} \stackrel{\text{def}}{=} \mathbf{HH}(A)^{\mathrm{Tate}}$

Proposition 3.2.8 (Morita Invariance)**.** *The trace maps* $\mathrm{tr} : M_r(A)^{\otimes n+1} \to A^{\otimes n+1}$ *induce an equivalence* $\mathbf{HH}(M_r(A)) \simeq \mathbf{HH}(A)$ *as complexes with S^1-action, and so induces equivalences on* $\mathbf{HC}, \mathbf{HC}^-, \mathbf{HP}$*. More generally, one can define* $\mathbf{HH}(\mathcal{C})$*, as complex with S^1-action up to weak equivalence, for dg-categories and show a strong form of Morita invariance.*

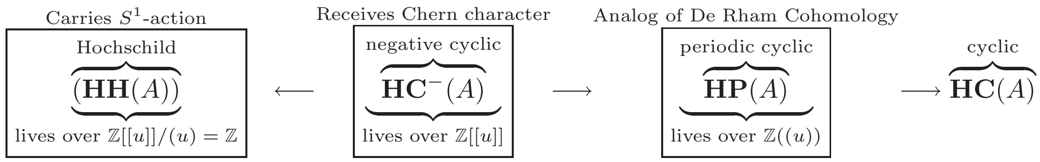
Construction 3.2.9. The n^{th} space of the cyclic set underlying $\mathbf{HH}(\mathcal{C})$ is

$$[n] \longmapsto \bigoplus_{X_n,\ldots,X_0\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X_n, X_{n-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_1, X_0) \otimes \mathrm{Hom}_{\mathcal{C}}(X_0, X_1)$$

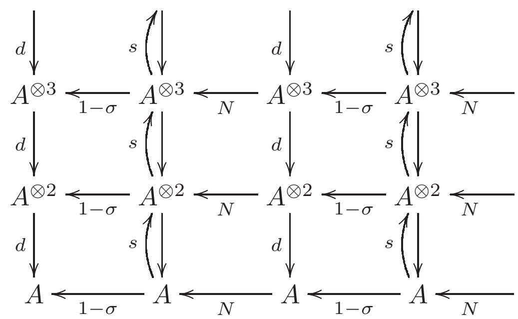
and the structure maps come from composition. Then, the formula immediately shows $\mathbf{HH}(A) = \mathbf{HH}(\mathcal{C})$ for $\mathcal{C}$ the full subcategory of A-mod corresponding to the object A: This is just the category with one object and A^{op} as automorphisms. The strong form of Morita invariance allows one to pass to thick closures (e.g., taking of finite direct sums and direct summands).

Remark 3.2.10. We can see Morita Invariance of $\mathbf{HC}$ (at the level of just homotopy groups) from e.g., a suitable application of the Gysin Sequence (Lemma 3.1.5) and Morita Invariance for $\mathbf{HH}$.

3.2.11. A heuristic diagram of relevant maps (and how things fit together):



Remark 3.2.12. There is another, more direct from a homological algebra point of view, way of getting a double-complex for $\mathbf{HC}(A)$. The idea is to mix, by hand, the standard resolution computing group homology of $\mathbb{Z}/(n+1)$ with two complexes (one of them $(\mathbf{HH}(A), d)$, the other contractible) related to $A^{\otimes(n+1)}$:



The even columns are the Hochschild complex with its d differential, but the odd columns admit a contracting homotopy s. Removing them and passing to "Adams-style" indexing (or is it back from?) gives the "familiar" complex for $\mathbf{HC}(A)$. In terms of this, $B = (1-\sigma)sN$.

This complex rise to a spectral sequence with E_1-page $H_{p-q}(B\mathbb{Z}/(q+1), A^{\otimes q+1}) \Rightarrow \pi_{p+q}\mathbf{HC}(A)$. In characteristic zero, the only non-zero terms are for $p = q$ and $H_0(B\mathbb{Z}/(q+1), A^{\otimes q+1}) = A^{\otimes q+1}_{\mathbb{Z}/(q+1)}$. In other words, the Hochschild differential d factors through the quotient and (in char. 0) the resulting quotient complex computes $\pi_*\mathbf{HC}(A)$.

Exercise 3.2.13. Extend this complex in the obvious way to the left, to obtain an analogue of the complex for $\mathbf{HP}$. Suppose we work over $\mathbb{Q}$. Explain why taking $\mathrm{Tot}^{\oplus}$ instead of $\mathrm{Tot}^{\prod}$ would be a bad idea. (Hint: What does each row compute?)

3.3. **Chern Character.**

3.3.1. The Dennis trace map, which Nick began describing last time, gives the top row of

$$\begin{array}{ccc} K_n(A) = \pi_n B\,\mathrm{GL}(A)^+ & \xrightarrow{\text{Dennis trace}} & HH_n(A) = \pi_n \mathbf{HH}(A) \\ \downarrow & & \uparrow \\ H_n(B\,\mathrm{GL}(A)) = \pi_* C_*(B\,\mathrm{GL}(A)) & \dashrightarrow & HC_n^-(A) = \pi_*\mathbf{HC}^-(A) \end{array}$$

We might expect it to come from a map of spectra, in which case since $\mathbf{HH}(A)$ is an $H\mathbb{Z}$-spectrum it would factor through the Hurewicz map in the left column. In the right column we have the natural map $\mathbf{HC}^-(A) = (\mathbf{HH}(A))^{hS^1} \to \mathbf{HH}(A)$.

Theorem 3.3.2. *The Dennis trace map does in fact factor through a map $H_n(B\,\mathrm{GL}(A)) \to HC_n^-(A)$.*

4. HKR Revisited

Filling out the analogies having to do with the "differential picture" (i.e., the first column):

Name	Symbol	Analogy
Hochschild chains	$\mathbf{HH}(A)$	Kahler modules
Connes' Differential	S^1 action B on $\mathbf{HH}(A)$	De Rham differential
Periodic cyclic chains	$\mathbf{HP}(A)$	($\mathbb{Z}/2$-graded) De Rham cohomology

4.1. De Rham differential, Connes differential.

Proposition 4.1.1. *The Connes B operator induces a map $B : HH_k(A) \to HH_{k+1}(A)$ which, under the HKR identification of Prop. 2.1.3, is (up to sign) the De Rham differential.*

4.2. Periodic cyclic Homology and the non-smooth case.

We noticed in the "table of analogies" that (homotopy groups of) $\mathbf{HP}(A)$ only saw the De Rham cohomology and did not let any part of the underlying De Rham *complex* seep out. This is crucial, because it allows the following to be true:

Theorem 4.2.1. *Suppose A is a commutative algebra (say finite-type over $\mathbb{C}$, but* not necessarily regular*). Then,*

$$HP_k(A) \simeq \prod_n H^{k+2n}_{sing}((\operatorname{Spec} A)(\mathbb{C}), \mathbb{C})$$

so that periodic cyclic homology recovers the "true" $\mathbb{Z}/2$-periodic cohomology.